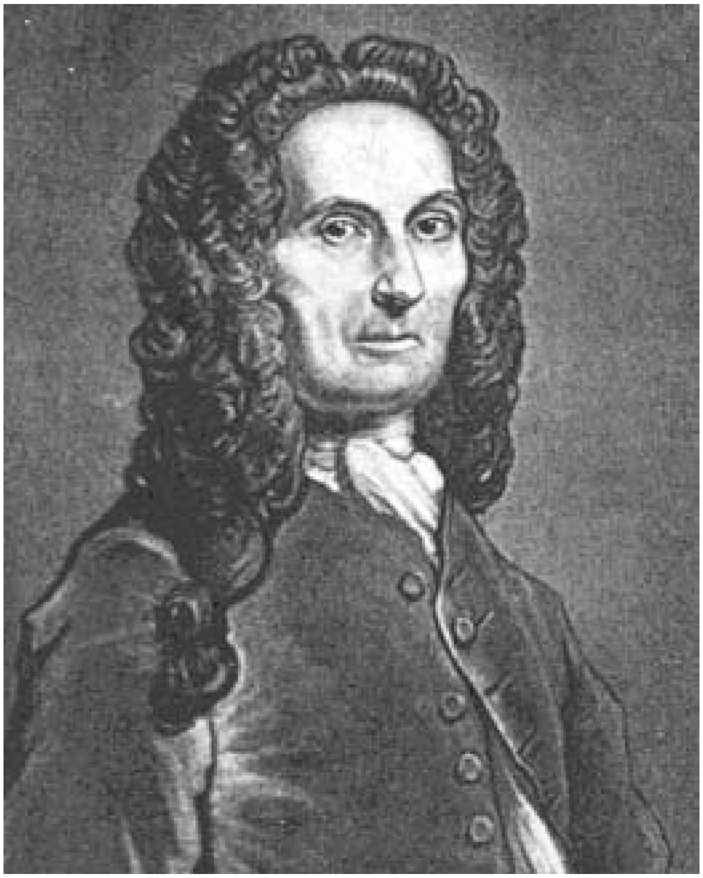


Abraham de Moivre

May 26, 1667 in Vitry-le-François, Champagne, France – November 27, 1754 in London, England) was a French mathematician famous for de Moivre's formula, which links complex numbers and trigonometry, and for his work on the normal distribution and probability theory. He was elected a Fellow of the Royal Society in 1697, and was a friend of Isaac Newton, Edmund Halley, and James Stirling.

The social status of his family is unclear, but de Moivre's father, a surgeon, was able to send him to the Protestant academy at Sedan (1678-82). de Moivre studied logic at Saumur (1682-84), attended the Collège de Harcourt in Paris (1684), and studied privately with Jacques Ozanam (1684-85). It does not appear that De Moivre received a college degree.

de Moivre was a Calvinist. He left France after the revocation of the Edict of Nantes (1685) and spent the remainder of his life in England.

Throughout his life he remained poor. It is reported that he was a regular customer of Slaughter's Coffee House, St. Martin's Lane at Cranbourn Street, where he earned a little money from playing chess.

He died in London and was buried at St Martin-in-the-Fields, although his body was later moved.

De Moivre wrote a book on probability theory, entitled The Doctrine of Chances. It is said in all seriousness that De Moivre correctly predicted the day of his own death. Noting that he was sleeping 15 minutes longer each day, De Moivre surmised that he would die on the day he would sleep for 24 hours. A simple mathematical calculation quickly yielded the date, November 27, 1754. He did indeed pass away on that day.

He first discovered the "closed form" expression for Fibonacci numbers linking the nth power of phi to the nth Fibonacci number.

He is also known for de Moivre's theorem which transfers a problem from complex numbers to trigonometry. You can derive many trigonometric identities by applying de Moivre's theorem.

$$r\left(\cos\theta + i\sin\theta\right)^{n} = r^{n}\left(\cos n\theta + i\sin n\theta\right)$$

$$r\left(\cos\theta + i\sin\theta\right)^{\frac{1}{n}} = r^{\frac{1}{n}}\left(\cos\frac{1}{n}\theta + i\sin\frac{1}{n}\theta\right)$$

Proof by Induction -

Base Case: $(cis\theta)^1 = \cos 1\theta + i\sin 1\theta$

Inductive Case:

[Must show $(cis\theta)^{k+1} = cis(k+1)\theta$ when $(cis\theta)^k = cis\ k\theta$ **]**

$(cis\theta)^{k+1} = (cis\theta)(cis\theta)^k = (cis\theta)cis(k\theta) =$

$\cos\theta\cos k\theta + i\cos\theta\sin k\theta + i\sin\theta\cos k\theta - \sin\theta\sin k\theta =$

$$(\cos\theta\cos k\theta - \sin\theta\sin k\theta) + i(\cos\theta\sin k\theta + \sin\theta\cos k\theta) =$$
$$\cos(k+1)\theta + i\sin(k+1)\theta = cis(k+1)\theta .$$

Since $\sin(x+y) = \sin x\cos y + \sin y\cos x$ and $\cos(x+y) = \cos x\cos y - \sin x\sin y$.

Introduction

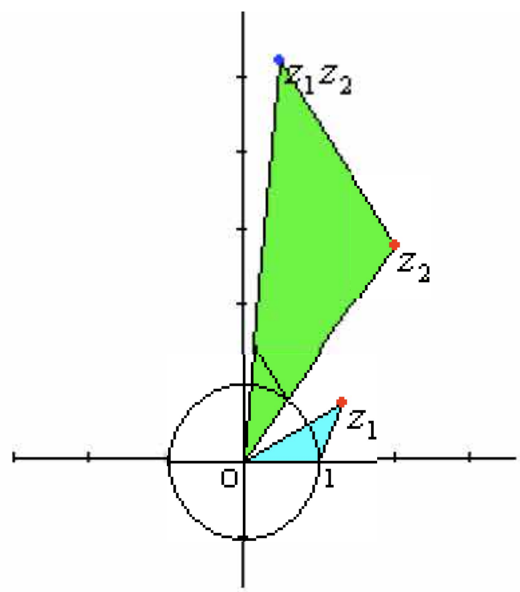
Multiplication of complex numbers will result in the rotation-enlargement of the factors.

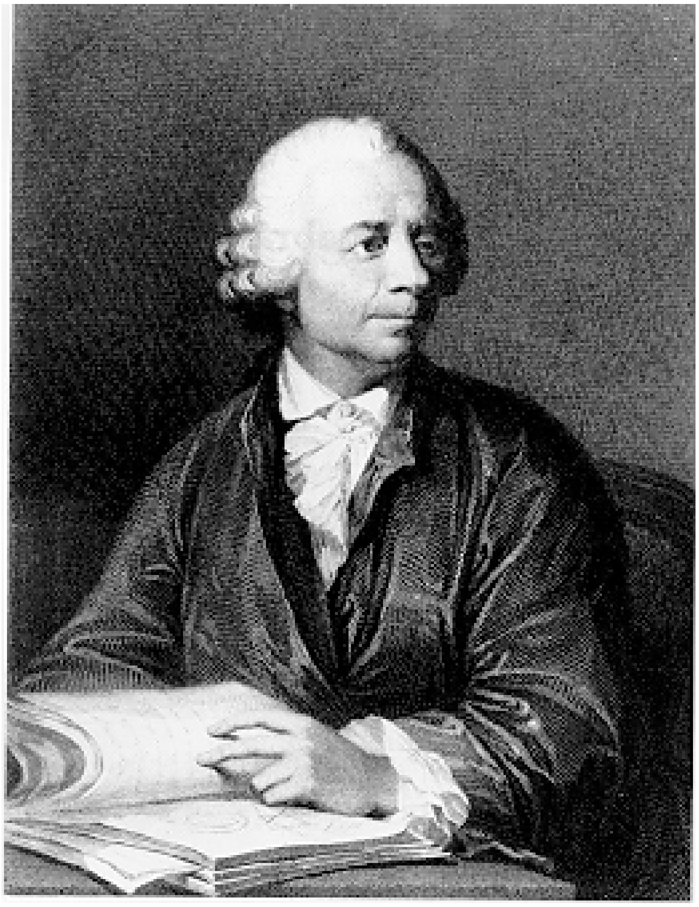
Multiplication by i will rotate its cofactor 90° or $\dfrac{\pi}{2}$ radians.

$$z = r e^{i\theta} = r(\cos\theta + i\sin\theta)$$

$$z_1 = r_1 e^{i\theta_1} \quad z_2 = r_2 e^{i\theta_2}$$

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$





Leonhard Euler

Leonhard Euler (1707-1783) was arguably the greatest mathematician of the eighteenth century (His closest competitor for that title is Lagrange) and one of the most prolific of all time; his publication list of 886 papers and books may be exceeded only by Paul Erdös. Euler's complete works fill about 90 volumes. Remarkably, much of this output dates from the the last two decades of his life, when he was totally blind.

Euler's important contributions were so numerous that terms like "Euler's formula" or "Euler's theorem" can mean many different things depending on context. Just in mechanics, one has Euler angles (to specify the orientation of a rigid body), Euler's theorem (that every rotation has an axis), Euler's equations for motion of fluids, and the Euler-Lagrange equation (that comes from calculus of variations). The "Euler's formula" with which most American calculus students are familiar defines the exponentials of imaginary numbers in terms of trigonometric functions. But there is another "Euler's formula" that (to use the modern terminology adopted long after Euler's death) gives the values of the Riemann zeta function at positive even integers in terms of Bernoulli numbers. There are both Euler numbers and Eulerian numbers, and they aren't the same thing. Euler's study of the bridges of Königsberg can be seen as the beginning of combinatorial topology (which is why the Euler characteristic bears his name).

Though born and educated in Basel, Switzerland, Euler spent most of his career in St. Petersburg and Berlin. He joined the St. Petersburg Academy of Sciences in 1727. In 1741 he went to Berlin at the invitation of Frederick the Great, but he and Frederick never got on well and in 1766 he returned to St. Petersburg, where he remained until his death. Euler's prolific output caused a tremendous problem of backlog: the St. Petersburg Academy continued publishing his work posthumously for more than 30 years. Euler married twice and had 13 children, though all but five of them died young.

Euler's powers of memory and concentration were legendary. He could recite the entire Aeneid word-for-word. He was not troubled by interruptions or distractions; in fact, he did much of his work with his young children playing at his feet. He was able to do prodigious calculations in his head, a necessity after he went blind. The contemporary French mathematician Condorcet tells the story of two of Euler's students who had independently summed seventeen terms of a complicated infinite series, only to disagree in the fiftieth decimal place; Euler settled the dispute by recomputing the sum in his head.

$$re^{i\theta} = r\left(\cos\theta + i\sin\theta\right), \qquad e^{\pi i} + 1 = 0$$

i

$i^2 = -1$

$i^3 = -i$

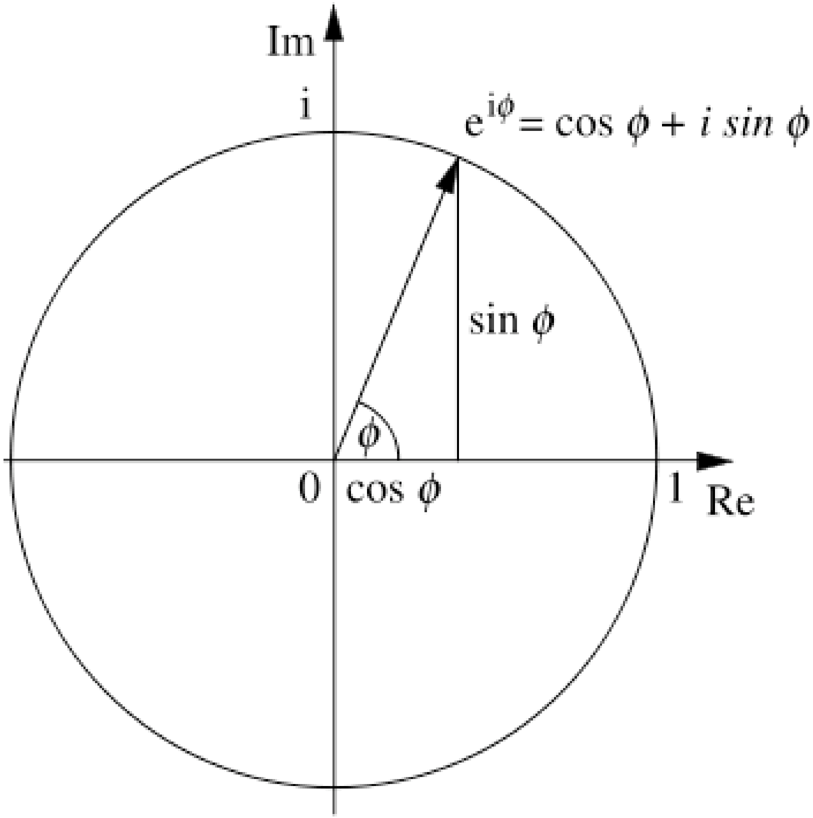
$i^4 = +1$

$i^5 = i$

$i^6 = -1$

$i^7 = -i$

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \cdots \\ &= 1 + ix - \frac{(x)^2}{2!} - \frac{i(x)^3}{3!} + \frac{(x)^4}{4!} + \frac{i(x)^5}{5!} - \frac{(x)^6}{6!} - \frac{i(x)^7}{7!} + \cdots \\ &= \left(1 - \frac{(x)^2}{2!} + \frac{(x)^4}{4!} - \frac{(x)^6}{6!} + \cdots\right) + i\left(x - \frac{(x)^3}{3!} + \frac{(x)^5}{5!} - \frac{(x)^7}{7!} + \cdots\right) \\ &= \cos x + i \sin x \end{aligned}$$



Jean-Robert Argand

(1768-1822) Jean-Robert Argand was an accountant and bookkeeper in Paris who was only an amateur mathematician. Little is known of his background and education. We do know that his father was Jacques Argand and his mother Eves Canac. In addition to his date of birth, the date on which he was baptized is known - 22 July 1768.

Among the few other facts known of his life is a little information about his children. His son was born in Paris and continued to live there, while his daughter, Jeanne-Françoise-Dorothée-Marie-Elizabeth Argand, married Félix Bousquet and they lived in Stuttgart.

Argand is famed for his geometrical interpretation of the complex numbers where i is interpreted as a rotation through 90degrees. The concept of the modulus of a complex number is also due to Argand but Cauchy, who used the term later, is usually credited as the originator this concept. The Argand diagram is taught to most school children who are studying mathematics and Argand's name will live on in the history of mathematics through this important concept. However, the fact that his name is associated with this geometrical interpretation of complex numbers is only as a result of a rather strange sequence of events.

The first to publish this geometrical interpretation of complex numbers was Caspar Wessel. The idea appears in Wessel's work in 1787 but it was not published until Wessel submitted a paper to a meeting of the Royal Danish Academy of Sciences on 10 March 1797. The paper was published in 1799 but not noticed by the mathematical community. Wessel's paper was rediscovered in 1895 when Juel draw attention to it and, in the same year, Sophus Lie republished Wessel's paper.

This is not as surprising as it might seem at first glance since Wessel was a surveyor. However, Argand was not a professional mathematician either, so when he published his geometrical interpretation of complex numbers in 1806 it was in a book which he published privately at his own expense. His knowledge of the book trade allowed him to put out this small edition but one would have expected it to be in a less noticable place than Wessel's work which after all was published by the Royal Danish Academy. Perhaps even more surprisingly, Argand's name did not even appear on the book so it was impossible to identify the author.

The way that Argand's work became known is rather complicated. Legendre was sent a copy of the work and he sent it to François Français although neither knew the identity of the author. After François Français's death in 1810 his brother Jacques Français worked on his papers and he discovered Argand's little book among them. In September 1813 Jacques Français published a work in which he gave a geometric representation of complex numbers, with interesting applications, based on Argand's ideas. Jacques Français might easily have claimed these ideas for himself, but he did quite the reverse. He ended his paper by saying that the idea was based on the work of an unknown mathematician and he asked that the mathematician should make himself known so that he might receive the credit for his ideas.

The article by Jacques Français appeared in Gergonne's journal Annales de mathématiques and Argand responded to Jacques Français's request by acknowledging that he was the author and submitting a slightly modified version of his original work with some new applications to the Annales de mathématiques. Now there is nothing like an argument to bring something to the attention of the world and this is exactly what happened next. A vigorous discussion between Jacques Français, Argand and Servois took place in the pages of Gergonne's Journal. In this correspondence Jacques Français and Argand argued in favour of the validity of the geometric representation, while Servois argued that complex numbers must be handled using pure algebra.

One might have expected that Argand would have made no other contributions to mathematics. However this is not so and, although he will always be remembered for the Argand diagram, his best work is on the fundamental theorem of algebra and for this he has received little credit. He gave a beautiful proof (with small gaps) of the fundamental theorem of algebra in his work of 1806, and again when he published his results in Gergonne's Journal in 1813. Certainly Argand was the first to state the theorem in the case where the coefficients were complex numbers. Petrova, in [6], discusses the early proofs of the fundamental theorem and remarks that Argand gave an almost modern form of the proof which was forgotten after its second publication in 1813.

After 1813 Argand did achieve a higher profile in the mathematical world. He published eight further articles, all in Gergonne's Journal, between 1813 and 1816. Most of these are based on either his original book, or they comment on papers published by other mathematicians. His final publication was on combinations where he used the notation (m, n) for the combinations of n objects selected from m objects.

In [1] Jones sums up Argand's work as follows:-

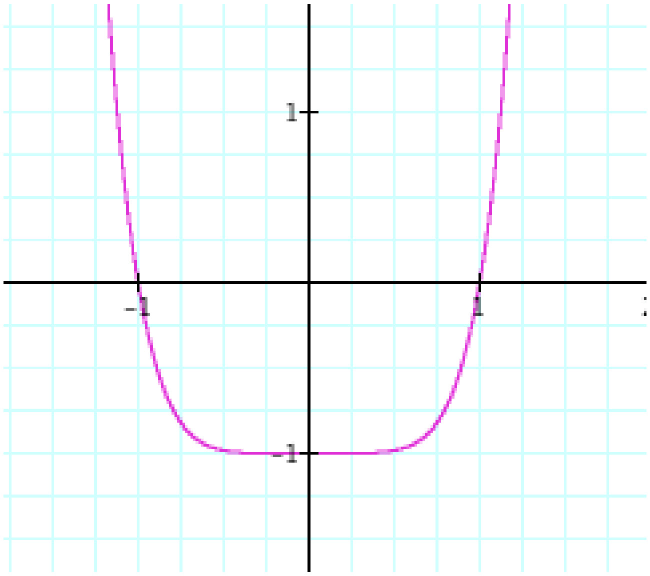
Argand was a man with an unknown background, a nonmathematical occupation, and an uncertain contact with the literature of his time who intuitively developed a critical idea for which the time was right. He exploited it himself. The quality and significance of his work were recognised by some of the geniuses of his time, but breakdowns in communication and the approximate simultaneity of similar developments by other workers force a historian to deny him full credit for the fruits of the concept on which he laboured.

Example

Find the 6 sixth roots of unity.

$$\sqrt[6]{1} = (\ 1 + 0i)^{\frac{1}{6}} = (1\ cis\ 0)^{\frac{1}{6}} =$$

$$1^{\frac{1}{6}}\left(cis\left(\theta + \frac{2\pi k}{n}\right)\right) \quad k\ = 1,2,3,4,5,6$$



$$= 1\left(cis\left(0 + \frac{2}{6}\pi\right)\right) = \cos\ \frac{1}{3}\pi + i\sin\ \frac{1}{3}\pi = \frac{1}{2} + \frac{1}{2}i\sqrt{3}$$

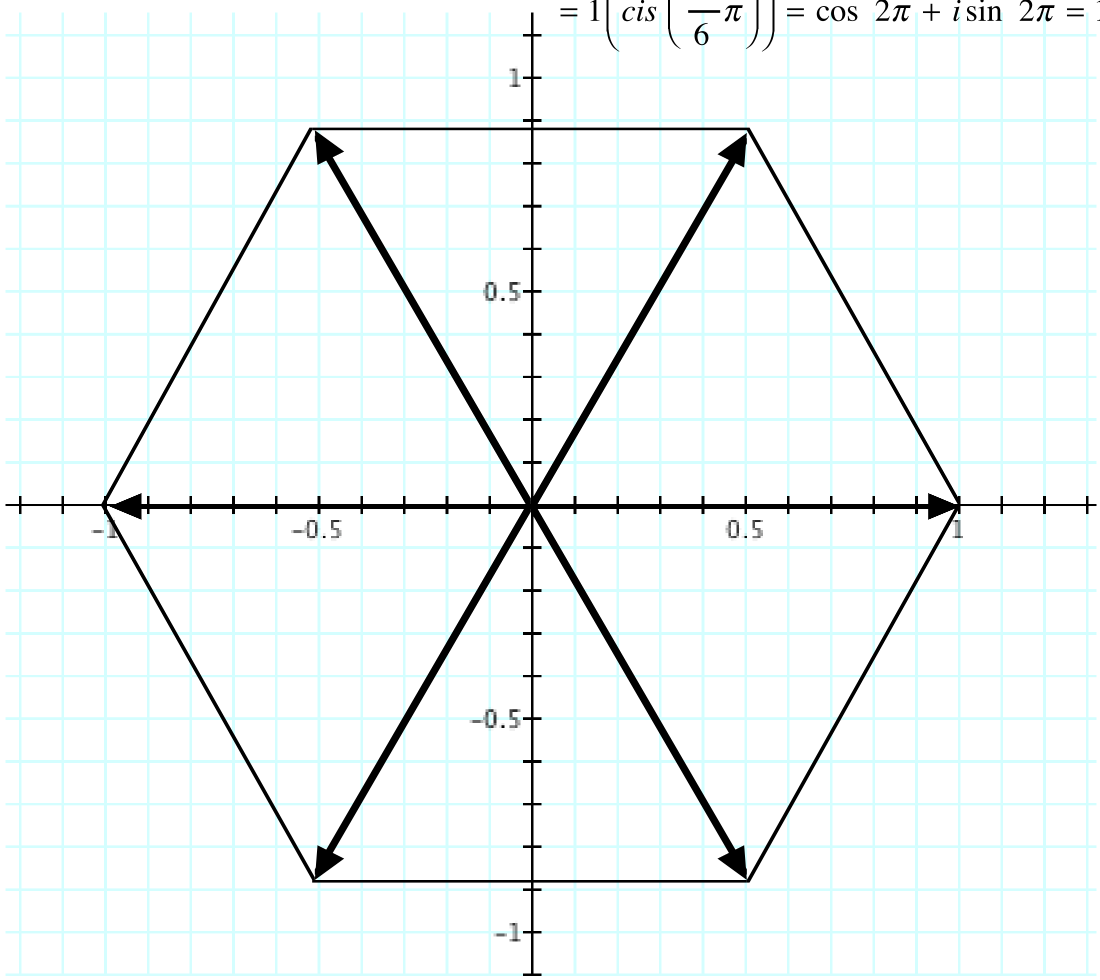
$$= 1\left(cis\left(\ \frac{4}{6}\pi\right)\right) = \cos\ \frac{2}{3}\pi + i\sin\ \frac{2}{3}\pi = -\frac{1}{2}\sqrt{3} + \frac{1}{2}i$$

$$= 1\left(cis\left(\frac{6}{6}\pi\right)\right) = \cos\ \pi + i\sin\ \pi = -1 + 0i = -1$$

$$= 1\left(cis\left(\ \frac{8}{6}\pi\right)\right) = \cos\ \frac{4}{3}\pi + i\sin\ \frac{4}{3}\pi = -\frac{1}{2}\sqrt{3} - \frac{1}{2}i$$

$$= 1\left(cis\left(\frac{10}{6}\pi\right)\right) = \cos\ \frac{5}{3}\pi + i\sin\ \frac{5}{3}\pi = \frac{1}{2} - \frac{1}{2}i\sqrt{3}$$

$$= 1\left(cis\left(\frac{12}{6}\pi\right)\right) = \cos\ 2\pi + i\sin\ 2\pi = 1 + 0i = 1$$



Some Exercises

Answer the following:

1. Perform the indicated operations

a) $2\sqrt{-216} + \dfrac{18\sqrt{2}}{\sqrt{-3}}$ **b)** $\left(-8 + \sqrt{-18}\right)\left(3 + 2\sqrt{-50}\right)$

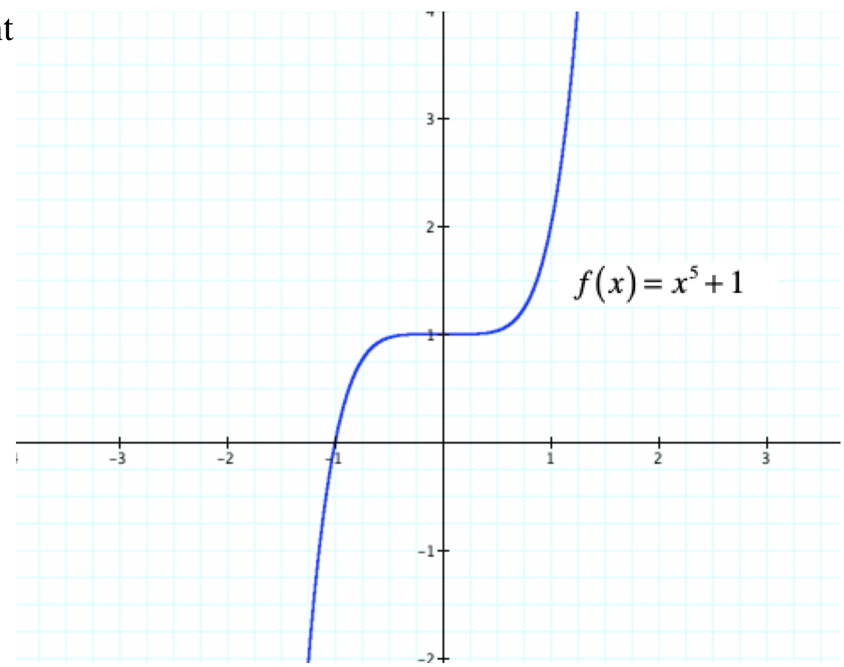
2. By what number must $6i$ be divided so that the quotient will be $2 + 2i\sqrt{2}$?

3. Represent graphically $-3 - 5i$, $8\ cis\ \dfrac{2\pi}{3}$, and $4e^{\frac{1}{3}\pi i}$.

4. Perform the indicated operations

a) $\dfrac{12\ cis\ \dfrac{17\pi}{12}}{3\ cis\ \dfrac{17\pi}{35}}$ **b)** $\left[\left(2\ cis\ 120^\circ\right)\right]^5$

5. Find the 5 fifth roots of -1 and represent them graphically.



6. a) $\ln i =$ **b)** $\log i =$ **c)** $i^i =$

True - False

1. An irrational number is imaginary.

2. The product of two positive imaginaries is a negative real number.

3. $\sqrt[3]{-2}$ is a pure imaginary number.

4. One of the cube roots of 27 is $-\frac{3}{2} - \frac{3}{2}\sqrt{-3}$.

5. $\sqrt{-2} \cdot \sqrt{-3} = \sqrt{6}$.

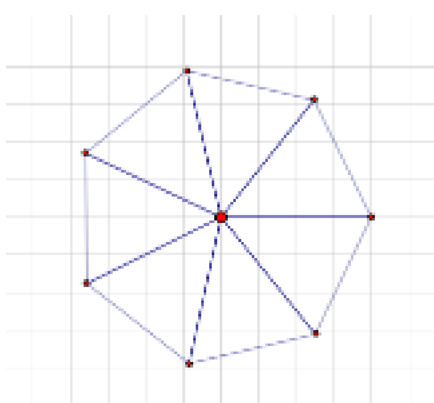
6. $\sqrt{-5} = 5i$.

7. $i^{33} = \sqrt{-1}$.

8. $\frac{\sqrt{-5}}{\sqrt{-3}} = +\frac{1}{3}\sqrt{15}$.

9. The product of the three cube roots of 1 is 1.

10. It is impossible to construct with ruler and compass the 7 seventh roots of 1.



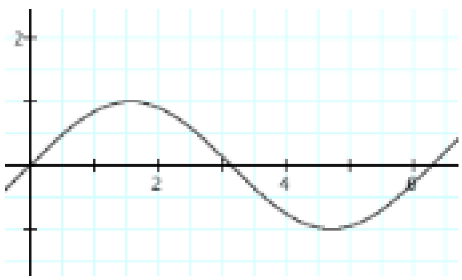
Notes on A-C and D-C Current

(1942)

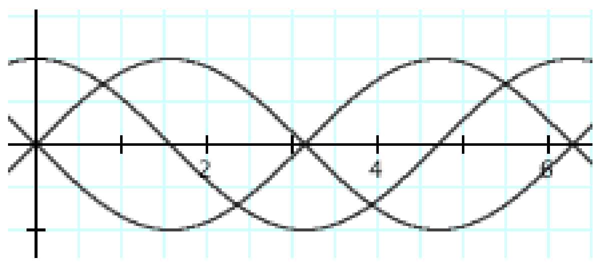
- A-C over long distances, push 212 kV vs 3 kV for D-C
- A-C easier to load level and ramp
- **No** D-C transformers, power comes from D-C generators in series
- A-C transformers have no moving parts, 95% efficient
- A-C induction motors are cheaper, easier to design, and require very little maintenance as compared to D-C generators
- A-C generators must be excited by direct current
- Most D-C generators are powered by A-C current

(2019)

- UHVAC 1000 kV vs UHVDC 800 kV



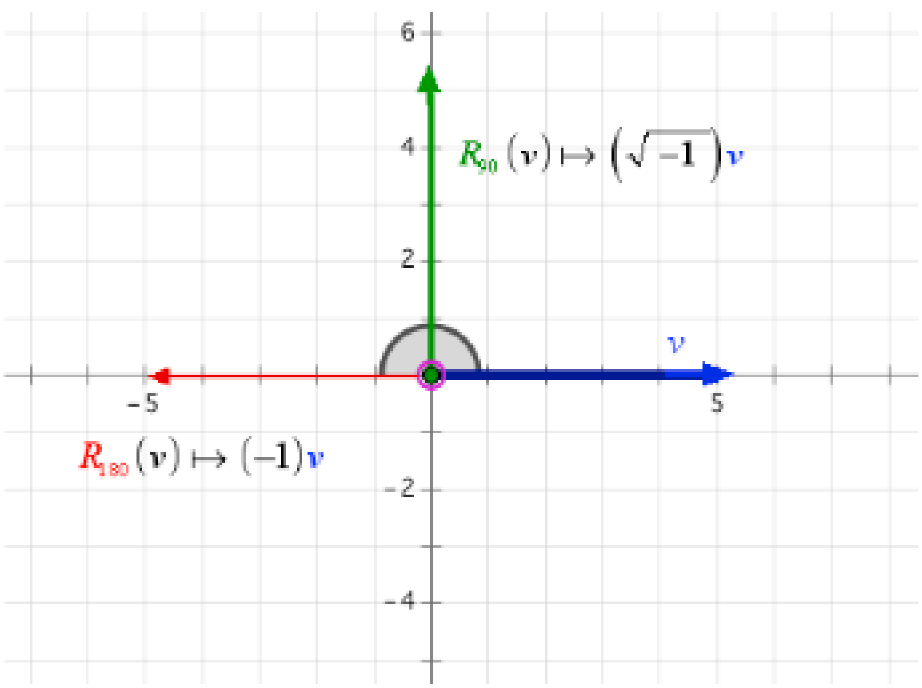
single phase voltage wave



polyphase voltage waves

$$v_1 = E_1 \sin \omega t$$

$$I = I_H + jI_V$$



(from *Alternating-Current Circuits* by Morecock 1942)

Example 4-5: Given the following value for the circuit below. determine

a) the line currents **b)** the phase voltages c) the line voltages

d) the alternator impedance voltages **e)** the line impedance voltages

Solve with phase sequence $E_{n'a'}$, $E_{n'b'}$, $E_{n'c'}$

$$E_{n'a'} = 2400 + j0 = 2400\angle 0^\circ \text{ volts}$$

$$E_{n'b'} = -1200 - j2080 = 2400\angle {}^{-}120^\circ \text{ volts}$$

$$E_{n'c'} = -1200 + j2080 = 2400\angle 120^\circ \text{ volts}$$

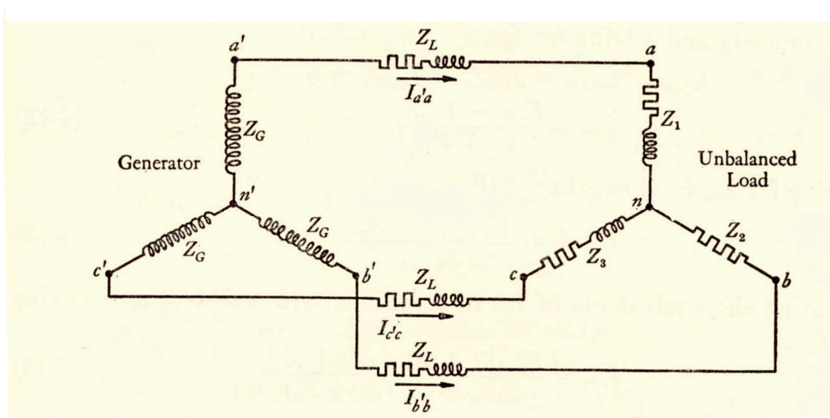
$$Z_G = 1 + j4 = 4.13\angle 75^\circ 58' \text{ ohms}$$

$$Z_L = 6 + j4 = 7.21\angle 33^\circ 42' \text{ ohms}$$

$$Z_1 = 34.7 + j20.0 = 40\angle 30^\circ \text{ ohms}$$

$$Z_2 = 50.0 + j0 = 50\angle 0^\circ \text{ ohms}$$

$$Z_3 = 42.4.7 + j42.4 = 60\angle 45^\circ \text{ ohms}$$



Solution.

(*a*)

$$Z_a = Z_G + Z_L + Z_1 = 41.7 + j\,28.0 = 50.3\underline{/33^\circ\ 51'}\ \text{ohms}$$
$$Z_b = Z_G + Z_L + Z_2 = 57.0 + j\,8.0 = 57.6\underline{/8^\circ}\ \text{ohms}$$
$$Z_c = Z_G + Z_L + Z_3 = 49.4 + j\,50.4 = 70.6\underline{/45^\circ\ 34'}\ \text{ohms}$$
$$E_{c'b'} = 1200 - j\,2080 - 1200 - j\,2080$$
$$= 0 - j\,4160 = 4160\overline{\backslash 90^\circ}\ \text{volts}$$
$$E_{b'a'} = 1200 + j\,2080 + 2400 - j\,0$$
$$= 3600 + j\,2080 = 4160\underline{/30^\circ}\ \text{volts}$$
$$Z_b + Z_c = 57.0 + j\,8.0 + 49.4 + j\,50.4$$
$$= 106.4 + j\,58.4 = 121.4\underline{/28^\circ\ 44'}\ \text{ohms}$$

Solving for the denominator of equations 4-14 and 4-15,

$$D = (57.0 + j\,8.0)(49.4 + j\,50.4) + (41.7 + j\,28.0)$$
$$(106.4 + j\,58.4) = 5217 + j\,8680 = 10{,}120\underline{/58^\circ\ 56'}\ \text{ohms}^2$$

Solving for $I_{a'a}$, $I_{b'b}$, and $I_{c'c}$ by substituting in equations 4-14, 4-15 and 4-16,

$$I_{a'a} = \frac{(4160\overline{\backslash 90^\circ})(57.6\underline{/8^\circ}) + (4150\underline{/30^\circ})(121.4\underline{/28^\circ\ 44'})}{10{,}120\underline{/58^\circ\ 56'}}$$
$$= \frac{353{,}500\underline{/33^\circ\ 17'}}{10{,}120\underline{/58^\circ\ 56'}} = 34.9\overline{\backslash 25^\circ\ 39'}\ \text{amp} \qquad \textit{Ans.}$$
$$I_{b'b} = \frac{(4160\overline{\backslash 90^\circ})(50.3\underline{/33^\circ\ 51'}) - (4160\underline{/30^\circ})(70.6\underline{/45^\circ\ 34'})}{10{,}120\underline{/58^\circ\ 56'}}$$
$$= \frac{45{,}900\overline{\backslash 84^\circ\ 36'}}{10{,}120\underline{/58^\circ\ 56'}} = 45.3\overline{\backslash 143^\circ\ 32'}\ \text{amp} \qquad \textit{Ans.}$$
$$I_{c'c} = -\,(34.9\overline{\backslash 25^\circ\ 39'}) - (45.3\overline{\backslash 143^\circ\ 32'})$$
$$= -\,31.4 + j\,15.1 + 36.4 + j\,26.9$$
$$= 5.0 + j\,42.0 = 42.3\underline{/83^\circ\ 13'}\ \text{amp} \qquad \textit{Ans.}$$

(*b*)

$$V_{an} = I_{a'a}Z_1 = (34.9\overline{\backslash 25^\circ\ 39'})(40\underline{/30^\circ})$$
$$= 1396\underline{/4^\circ\ 21'} = 1391 + j\,106\ \text{volts} \qquad \textit{Ans.}$$
$$V_{bn} = I_{b'b}Z_2 = (45.3\overline{\backslash 143^\circ\ 32'})(50\underline{/0})$$
$$= 2265\overline{\backslash 143^\circ\ 32'} = -\,1820 - j\,1343\ \text{volts} \qquad \textit{Ans.}$$
$$V_{cn} = I_{c'c}Z_3 = (42.3\underline{/83^\circ\ 13'})(60\underline{/45^\circ})$$
$$= 2538\underline{/128^\circ\ 13'} = -\,1568 + j\,1992\ \text{volts} \qquad \textit{Ans.}$$

(*c*)

$$V_{ab} = V_{an} - V_{bn} = (1391 + j\,106) - (-\,1820 - j\,1343)$$
$$= 3211 + j\,1449 = 3520\underline{/24^\circ\ 17'}\ \text{volts} \qquad \textit{Ans.}$$
$$V_{bc} = V_{bn} - V_{cn} = (-\,1820 - j\,1343) - (-\,1568 + j\,1992)$$
$$= -\,252 - j\,3335 = 3344\overline{\backslash 94^\circ\ 19'}\ \text{volts} \qquad \textit{Ans.}$$
$$V_{ca} = V_{cn} - V_{an} = (-\,1568 + j\,1992) - (1391 + j\,106)$$
$$= -\,2959 + j\,1886 = 3509\underline{/147^\circ\ 28'}\ \text{volts} \qquad \textit{Ans.}$$

(*d*)

$$I_{a'a}Z_G = (34.9\overline{\backslash 25^\circ\ 39'})(4.13\underline{/75^\circ\ 58'})$$
$$= 144\underline{/50^\circ\ 19'}\ \text{volts} \qquad \textit{Ans.}$$
$$I_{b'b}Z_G = (45.3\overline{\backslash 143^\circ\ 32'})(4.13\underline{/75^\circ\ 58'})$$
$$= 187\overline{\backslash 67^\circ\ 34'}\ \text{volts} \qquad \textit{Ans.}$$
$$I_{c'c}Z_G = (42.3\underline{/83^\circ\ 13'})(4.13\underline{/75^\circ\ 58'})$$
$$= 174.5\underline{/159^\circ\ 11'}\ \text{volts} \qquad \textit{Ans.}$$

(*e*)

$$I_{a'a}Z_L = (34.9\overline{\backslash 25^\circ\ 39'})(7.21\underline{/33^\circ\ 42'})$$
$$= 252\underline{/8^\circ\ 3'}\ \text{volts} \qquad \textit{Ans.}$$
$$I_{b'b}Z_L = (45.3\overline{\backslash 143^\circ\ 32'})(7.21\underline{/33^\circ\ 42'})$$
$$= 326\overline{\backslash 109^\circ\ 50'}\ \text{volts} \qquad \textit{Ans.}$$
$$I_{c'c}Z_L = (42.3\underline{/83^\circ\ 13'})(7.21\underline{/33^\circ\ 42'})$$
$$= 305\underline{/116^\circ\ 55'}\ \text{volts} \qquad \textit{Ans.}$$