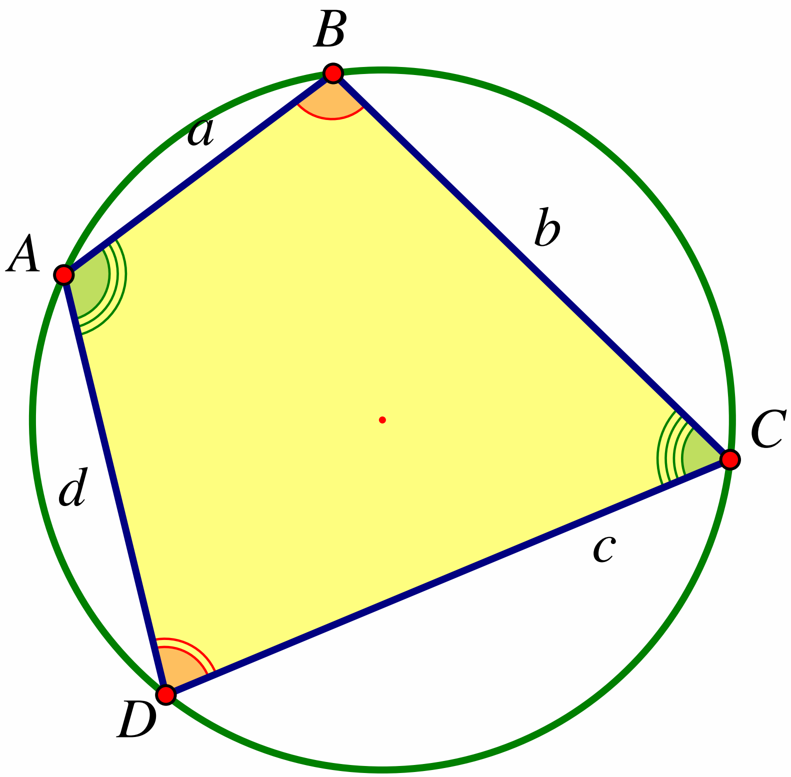
Theorem : In any cyclic quadrilateral , opposite angles are supplementary.



$m\angle ADC = 81.02°$
$m\angle ABC = 98.98°$
$m\angle ADC + m\angle ABC = 180.00°$

$m\angle DAB = 113.14°$
$m\angle DCB = 66.86°$
$m\angle DAB + m\angle DCB = 180.00°$

***Proof* :**

Construct diagonals $\overline{AC}$ and $\overline{BD}$.

$\angle BAC = \angle BDC$ and $\angle \text{BCA} = \angle ADB$

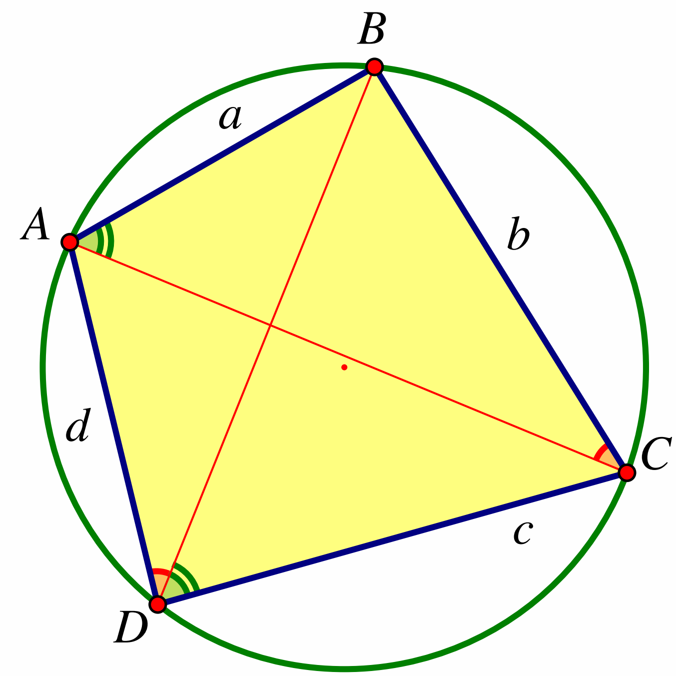
So then, $\angle BAC + \angle \text{BCA} = \angle BDC + \angle ADB = \angle ADC$
adding $\angle ABC$ to both sides we get

$\angle BAC + \angle \text{BCA} + \angle ABC = \angle ADC + \angle ABC$ or

$180° = \angle ADC + \angle ABC$ and also

$\angle BAD + \angle \text{BCD} = 360° - (\angle ADC + \angle ABC)$

$\therefore \angle BAD + \angle \text{BCD} = 180°$



Theorem : If the opposite angles in a quadrilateral are supplementary, the the quadrilateral is cyclic.

***Proof* :**

Construct diagonal $\overline{AC}$

Case # 1: D is interior to the $\odot$. This would mean that $\angle ADC > \frac{1}{2}\overset{\frown}{ABC}$ and then that

$\angle ADC + \angle ABC > 180°$ →←

Case # 2: D is exterior to the $\odot$. Then

$\angle ADC = \frac{1}{2}\overset{\frown}{ABC} - \frac{1}{2}\overset{\frown}{PQ}$ or that

$\frac{1}{2}\overset{\frown}{PQ} = \frac{1}{2}\overset{\frown}{ABC} - \angle ADC$ and

$\overset{\frown}{PQ} = \overset{\frown}{ABC} - 2\angle ADC$. Now

$\angle ABC = 180° - \angle AEC$ or $\angle ABC = 180° - \frac{1}{2}\overset{\frown}{ABC}$ or

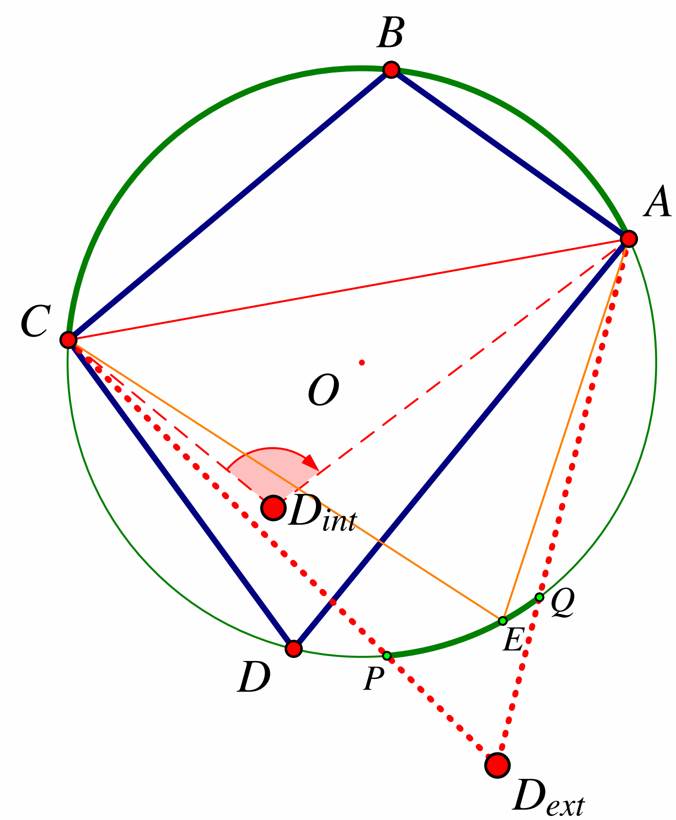
$\overset{\frown}{ABC} = 360° - 2\angle ABC$ so then

$\overset{\frown}{PQ} = 360° - 2\angle ABC - 2\angle ADC \Rightarrow$

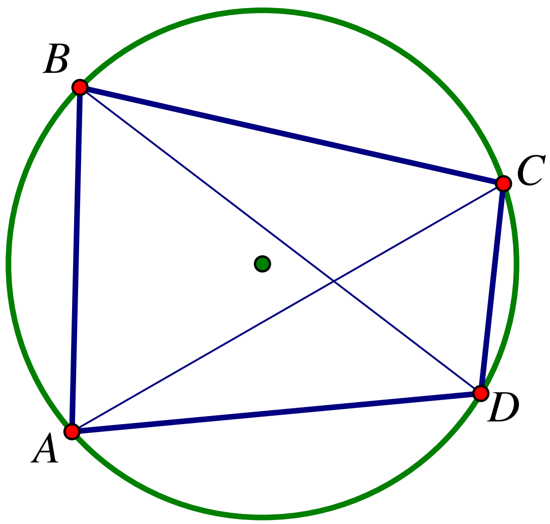
$\overset{\frown}{PQ} = 360° - 2(\angle ABC - \angle ADC)$

but $\overset{\frown}{PQ} \neq 0°$, so $\angle ABC - \angle ADC < 180°$. →←

$\therefore$ D must be on $\odot O$.



Theorem (Claudius Ptolemaeus) : In any cyclic quadrilateral, the product of the diagonals is equal to the sum of the products of the opposite sides. $\overline{BD} \cdot \overline{AC} \cong (\overline{AB} \cdot \overline{CD}) + (\overline{BC} \cdot \overline{DA})$



$m\,\overline{BD} = 9.53\ cm$

$m\,\overline{AC} = 9.41\ cm$

$m\,\overline{BD} \cdot m\,\overline{AC} = 89.65\ cm^2$

$m\,\overline{AB} = 6.52\ cm$

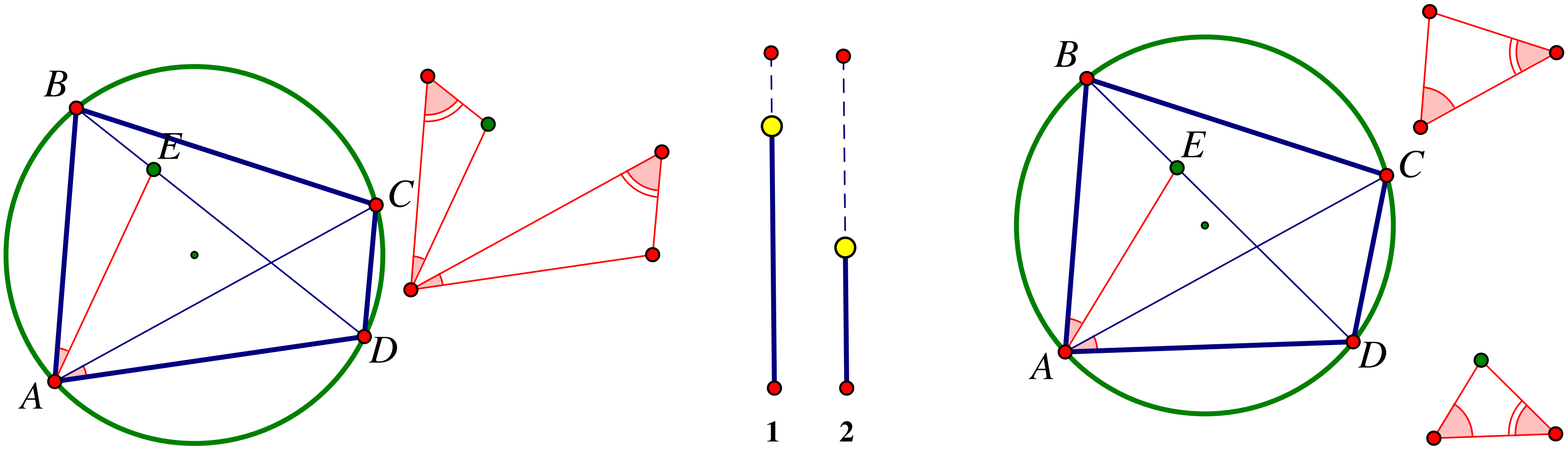
$m\,\overline{BC} = 8.20\ cm$

$m\,\overline{CD} = 3.99\ cm$

$m\,\overline{DA} = 7.76\ cm$

$(m\,\overline{AB} \cdot m\,\overline{CD}) + (m\,\overline{BC} \cdot m\,\overline{DA}) = 89.65\ cm^2$

Theorem (Ptolemy) : In any cyclic quadrilateral, the product of the diagonals is equal to the sum of the products of the opposite sides. $\overline{BD}\cdot\overline{AC} \cong (\overline{AB}\cdot\overline{CD}) + (\overline{BC}\cdot\overline{DA})$



Draw $\overline{AE} \ni \angle BAE = \angle CAD$, $\angle ABD = \angle ACD$, $\therefore \triangle ABE \sim \triangle ACD$ and $\dfrac{\overline{AB}}{\overline{AC}} = \dfrac{\overline{BE}}{\overline{CD}}$

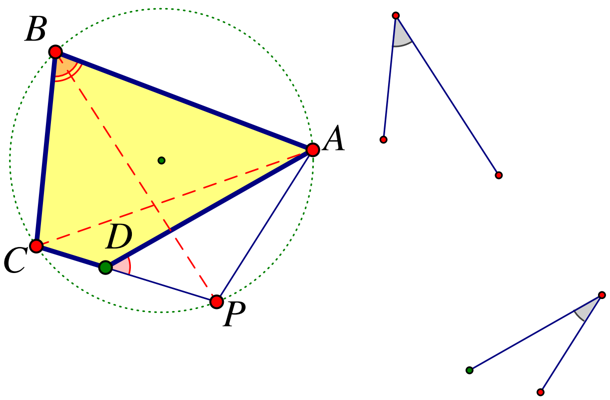
or $\overline{AB}\cdot\overline{CD} = \overline{AC}\cdot\overline{BE}.$ $\angle DEA$ and $\angle BEA$ are supplementary and also $\angle ABC$ and $\angle ADC$.

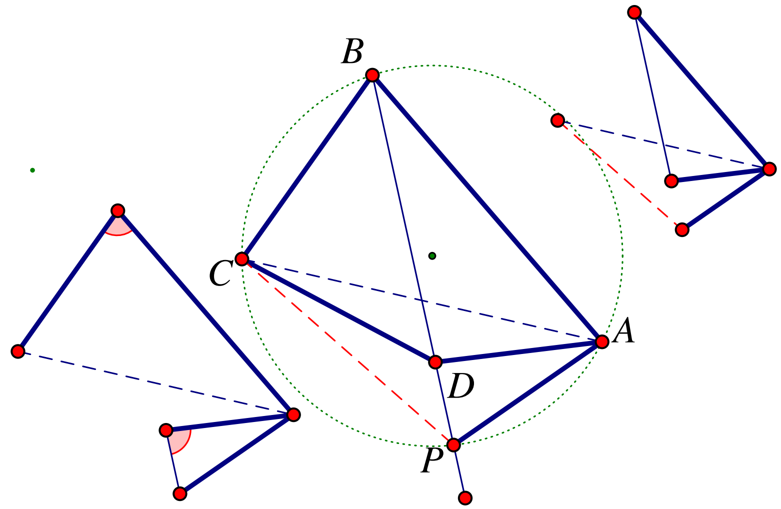
From the previous similar triangles $\angle BEA = \angle ADC$, so $\angle DEA = \angle ABC$. Also

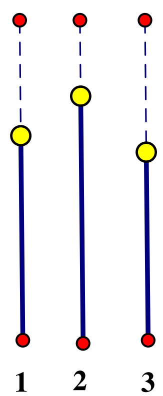
$\angle BCA = \angle BDA$, $\therefore \triangle ABC \sim \triangle AED$, and $\dfrac{\overline{AC}}{\overline{AD}} = \dfrac{\overline{BC}}{\overline{ED}}$, or $\overline{AC}\cdot\overline{ED} = \overline{AD}\cdot\overline{BC}$. Since

$\overline{BE} + \overline{ED} = \overline{BD}$, we have $\overline{AB}\cdot\overline{CD} + \overline{AD}\cdot\overline{BC} = \overline{AC}\cdot\overline{BE} + \overline{AC}\cdot\overline{ED} = \overline{AC}\cdot\overline{BD}$.

Theorem (Converse) : If the product of the diagonals of a quadrilateral is equal to the sum of the products of the opposite sides $\overline{BD} \cdot \overline{AC} \cong (\overline{AB} \cdot \overline{CD}) + (\overline{BC} \cdot \overline{DA})$, then the quadrilateral is cyclic







Assume $ABCD$ is **not cyclic** and let P be such that $ABCP$ is cyclic.

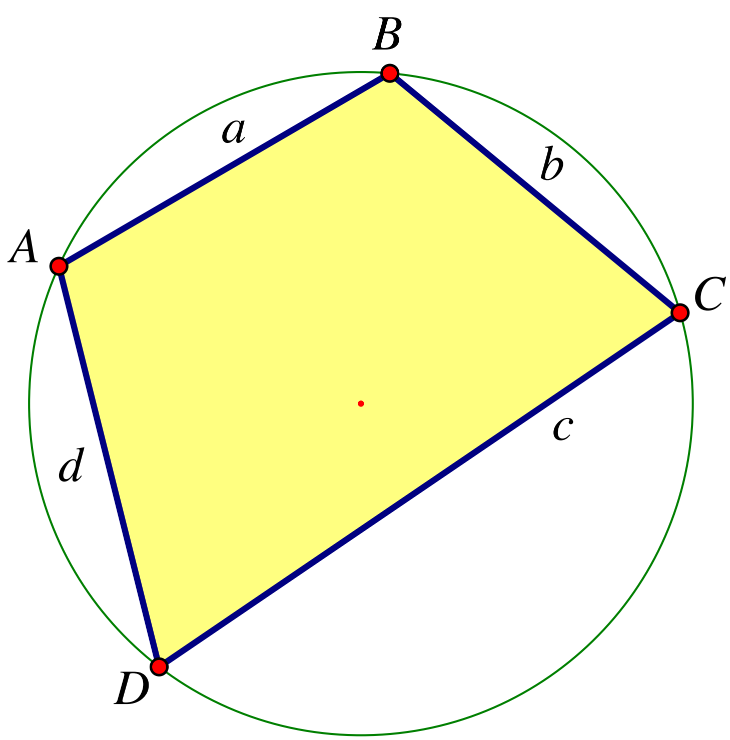
- Case C, D, and P collinear: Then $\angle ADP \neq \angle ABC$
- Case C, D, and P not collinear: Then it is possible that $\angle ADP = \angle ABC$

but then $\overline{CP} < \overline{CD} + \overline{DP}$ and since $\overline{DP} = \dfrac{\overline{AD} \cdot \overline{BC}}{\overline{AB}}$ and $\overline{CP} = \dfrac{\overline{AC} \cdot \overline{BD}}{\overline{AB}}$

then $\overline{AC} \cdot \overline{BD} < \left(\overline{AB} \cdot \overline{CD}\right) + \left(\overline{AD} \cdot \overline{BC}\right)$

Theorem (Brahmagupta) : In any cyclic quadrilateral

$$\text{Area}(ABCD) = \sqrt{(s- a)(s - b)(s - \text{c})(s - d)} \text{ where } s = \frac{1}{2}(a + b + c + d).$$



$a = 6.54\ cm$
$b = 6.43\ cm$
$c = 10.76\ cm$
$d = 7.06\ cm$

$s = 15.39\ cm$

$\sqrt{(s-a)\cdot(s-b)\cdot(s-c)\cdot(s-d)} = 55.35\ cm^2$

$Area\ ABCD = 55.35\ cm^2$

***Proof* :**

Construct diagonal $\overline{BD}$.

$$A_{ABCD} = A_{\Delta ABD} + A_{\Delta BCD}$$
$$= \frac{1}{2}a(d\sin\angle A) + \frac{1}{2}b(c\sin\angle(\pi - A))$$
$$= \frac{1}{2}(ad + bc)\sin\angle \mathrm{A}$$

Now, $l^2 = a^2 + d^2 - 2ad\cos\angle A$ and
$l^2 = b^2 + c^2 - 2bc\cos(\pi - A)$

$$\Rightarrow \cos\angle A = \frac{a^2 + d^2 - b^2 + c^2}{2(ad + bc)}$$

$$\therefore \quad A_{ABCD} = \frac{1}{2}(ad + bc)\sqrt{1 - (\frac{a^2 + d^2 - b^2 + c^2}{2(ad + bc)})^2}$$ since
$\sin\angle A = \sqrt{1 - \cos^2\angle A}$

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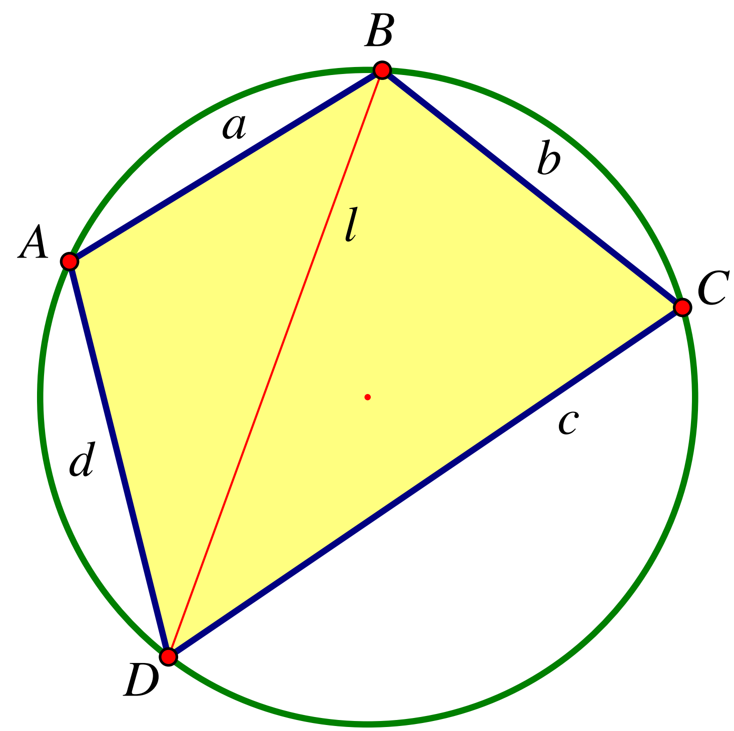
$$A_{ABCD} = \sqrt{(s - a)(s - b)(s - \mathrm{c})(s - d)}$$

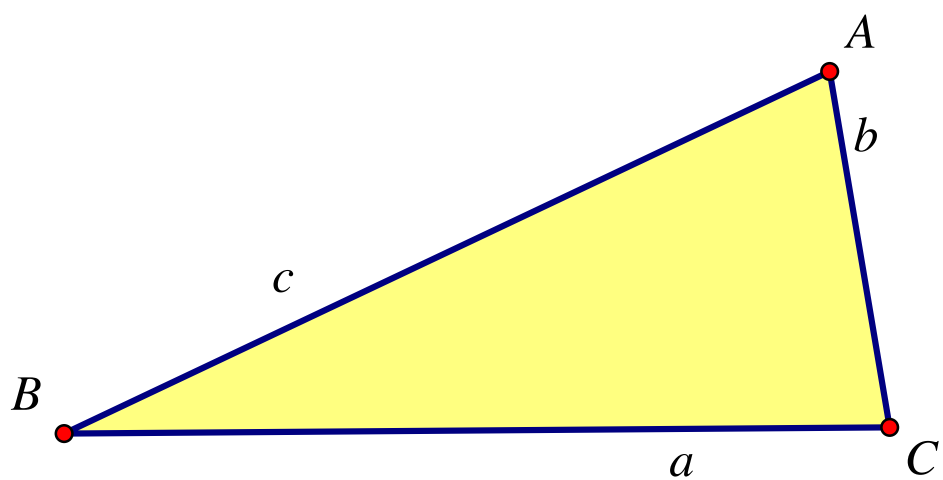


Theorem (Heron) : In any triangle,

$$\text{Area}(\Delta ABC) = \sqrt{s(s-a)(s-b)(s-c)} \text{ where } s = \frac{1}{2}(a+b+c).$$

$a = 14.08$ cm
$b = 6.15$ cm
$c = 14.44$ cm

$s = 17.33$ cm

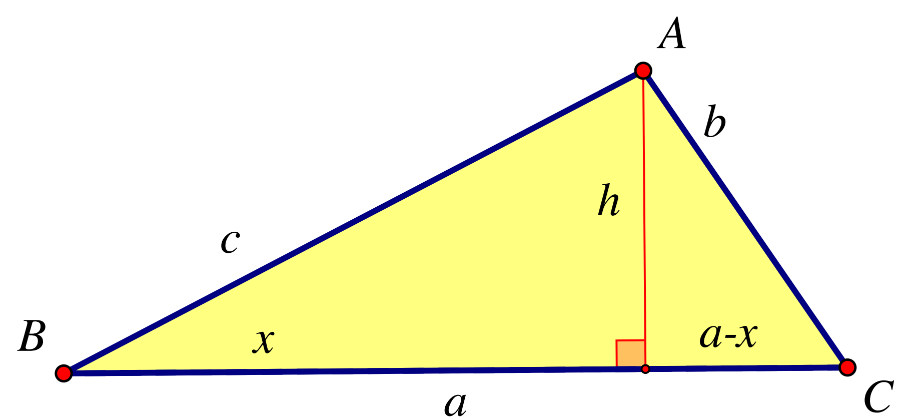


$$\sqrt{s \cdot (s-b) \cdot (s-c) \cdot (s-a)} = 42.76 \text{ cm}^2$$

$$\text{Area } \Delta CAB = 42.76 \text{ cm}^2$$

Theorem (Heron) : In any triangle,

$$\text{Area}(\Delta ABC) = \sqrt{s(s-a)(s-b)(s-c)} \text{ where } s = \frac{1}{2}(a+b+c).$$



***Proof* :**

Construct $h \perp a$. Now $h^2 = c^2 - x^2$ and $h^2 = b^2 - (a-x)^2$,

$\Rightarrow x = \dfrac{a^2 + c^2 - b^2}{2a}$ Also, since $h^2 = (c-x)(c+x)$

$$h^2 = \frac{(b-a+c)(b+a-c)(a+c-b)(a+c+b)}{4a^2}$$

Let $2s = a+b+c$, or $4a^2h^2 = 16s(s-a)(s-b)(s-c)$

$\Rightarrow \dfrac{1}{2}ah = \sqrt{s(s-a)(s-b)(s-c)}$

or $A = \sqrt{s(s-a)(s-b)(s-c)}$