Subject: Linear Algebra
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MATRICES AND MATRIX OPERATIONS

SIZE OF THE MATRIX is defined by number of rows and columns in the matrix. For the matrix that have m rows and n columns we say the size of the matrix is m x n. If matrix have the same number of rows (n) and columns (n), we call that matrix the squared (nxn) matrix.

column ↓ $\begin{bmatrix} 4 & 0 & 2 \\ 1 & 1 & 3 \\ 5 & 2 & 7 \\ 8 & 9 & 1 \end{bmatrix}_{4x3}$ $\begin{bmatrix} 1 & 4 & 7 \\ 8 & 6 & 2 \end{bmatrix}_{2x3}$ $\begin{bmatrix} 7 & 2 & 4 \\ 2 & 3 & 5 \\ 4 & 8 & 9 \end{bmatrix}_{3x3}$ $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{mxn}$

row →

squared 3x3 matrix

SPECIAL MATRICES

1) *Zero Matrix* – matrix that has all elements equal to 0; The notation for this matrix is O.
2) *Identity Matrix* – matrix that has all 1's on the diagonal; The notation for this matrix is I, but in some books, it can be E.

O = [0], $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, etc. I = [1], $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, etc.

ADDITION AND SUBTRACTION – We can add or subtract only matrices that are <u>same sizes</u>.

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 8 & 6 & 2 \end{bmatrix}_{2x3} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2x3} \quad C = \begin{bmatrix} 4 & 0 & 2 \\ 1 & 1 & 3 \\ 5 & 2 & 7 \\ 8 & 9 & 1 \end{bmatrix}_{4x3}$$

Matrices A and B are the same sizes, because they both have 2 rows and 3 columns, matrix C has the different size. So we can only do addition or subtraction with matrices A and B. For example, we can do A – B.

$$A - B = \begin{bmatrix} 1 & 4 & 7 \\ 8 & 6 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1-1 & 4-2 & 7-3 \\ 8-4 & 6-5 & 2-6 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 4 \\ 4 & 1 & -4 \end{bmatrix}$$

SCALAR MULTIPLES – If A is the matrix and c is the scalar (any number) then cA (this is the same as c x A) is the matrix that we get when we multiply <u>each</u> entry of the matrix A with the scalar c.

$$a)\ 3A = 3\begin{bmatrix} 7 & 2 & 4 \\ 2 & 3 & 5 \\ 4 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 3x7 & 3x2 & 3x4 \\ 3x2 & 3x3 & 3x5 \\ 3x4 & 3x8 & 3x9 \end{bmatrix} = \begin{bmatrix} 21 & 6 & 12 \\ 6 & 9 & 15 \\ 12 & 24 & 27 \end{bmatrix}$$

$$b)\ -1/2A = -1/2\begin{bmatrix} 7 & 2 & 4 \\ 2 & 3 & 5 \\ 4 & 8 & 9 \end{bmatrix} = \begin{bmatrix} -7/2 & -1 & -2 \\ -1 & -3/2 & -5/2 \\ -2 & -4 & -9/2 \end{bmatrix}$$

TRACE OF A MATRIX – The sum of the elements along the main diagonal of a square matrix. The trace of A is $21 + 9 + 27 = 57$.

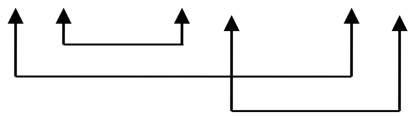
$$A = \begin{bmatrix} 21 & 6 & 12 \\ 6 & 9 & 15 \\ 12 & 24 & 27 \end{bmatrix}$$

MATRICES AND MATRIX OPERATIONS

MULTIPLYING MATRICES – We can multiply only matrices where the first matrix has the number of columns same as the number of rows of the second matrix. And new matrix AB will have same number of rows as the first matrix, and same number of columns as the second matrix. The next drawing will help you to understand.

A B = AB

m x r r x n = m x n



Note: A x B = AB and B x A = BA, but when we are multiplying matrices AB isn't the same as BA.

!!! AB ≠ BA

Can we multiply next matrices?

1) $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 0 & 2 \\ 1 & 1 & 3 \\ 5 & 2 & 7 \\ 8 & 9 & 1 \end{bmatrix}$

A B = AB

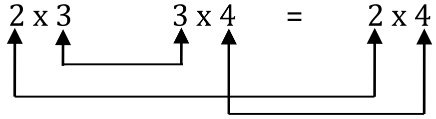
2 x 3 4 x 3 = ?

Matrix A has size 2x3 (2 rows and 3 columns), and matrix B has size 4x3 (4 rows and 3 columns). The number of columns of the matrix A is 3, and the number of the rows of the matrix B is 4, as these numbers are not the same we CAN'T multiply these two matrices.

2) $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$

A B = AB

2 x 3 3 x 4 = 2 x 4



Matrix A has size 2x3 (2 rows and 3 columns), and matrix B has size 3x4 (3 rows and 4 columns). The number of columns of the matrix A is 3, and the number of the rows of the matrix B is 3, as these numbers are the same we CAN multiply these two matrices.

Now, when we know that we can multiply A and B, we need to see what is going to be the size of the product matrix AB. Number of the rows of the matrix AB is equal to the number of the rows of the matrix A, it is 2. Number of the column of the matrix AB is equal to the number of the columns of the matrix B, it is 4. Thus, the size of the matrix AB is 2 x 4.

MATRICES AND MATRIX OPERATIONS

Now, we know that we can multiply these matrices, but how do we multiply matrices? We multiply each row of the first matrix with each column of the second matrix and put values in the specific order.

$$\text{AB} = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 1st\ row\ x\ 1st\ col & 1st\ r\ x\ 2nd\ c & 1st\ r\ x\ 3rd\ c & 1st\ r\ x\ 4th\ c \\ 2nd\ row\ x\ 1st\ col & 2nd\ r\ x\ 2nd\ c & 2nd\ r\ x\ 3rd\ c & 2nd\ r\ x\ 4th\ c \end{bmatrix}$$

How do we multiply row with column? The best way to explain this is with an example.

We are going to multiply the 1st row of the matrix A with the 1st column of the matrix B.

The first row of A is [1 2 4], the first column of B is $\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$. We are multiplying first number of the row, with the first number of the column, second number of the row with the second number of the column and third number of the row with the third number of the column. When we have these three numbers we are going to add them, and their sum will be the number we are going to put in the first row and in the first column of the matrix AB.

1st row x 1st column: 1x4 + 2x0 + 4x2 = 4 + 0 + 8 = 12. Now we have AB = $\begin{bmatrix} 12 & \square & \square & \square \\ \square & \square & \square & \square \end{bmatrix}$.

1st row x 2nd column: 1x1 + 2x(-1) + 4x7 = 1 -2 + 28 = 27 => AB = $\begin{bmatrix} 12 & 27 & \square & \square \\ \square & \square & \square & \square \end{bmatrix}$

1st row x 3rd column: 1x4 + 2x3 + 4x5 = 4 + 6 + 20 = 30 => AB = $\begin{bmatrix} 12 & 27 & 30 & \square \\ \square & \square & \square & \square \end{bmatrix}$

1st row x 4th column: 1x3 + 2x1 + 4x2 = 3 + 2 + 8 = 13 => AB = $\begin{bmatrix} 12 & 27 & 30 & 13 \\ \square & \square & \square & \square \end{bmatrix}$

2nd row x 1st column: 2x4 + 6x0 + 0x2 = 8 + 0 + 0 = 8 => AB = $\begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & \square & \square & \square \end{bmatrix}$

2nd row x 2nd column: 2x1 + 6x(-1) + 0x7 = 2 -6 + 0 = -4 = > AB = $\begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & \square & \square \end{bmatrix}$

2nd row x 3rd column: 2x4 + 6x3 + 0x5 = 8 + 18 + 0 = 26 => AB = $\begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & \square \end{bmatrix}$

2nd row x 1st column: 2x3 + 6x1 + 0x2 = 6 + 6 + 0 = 12 => AB = $\begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$

Rules for multiplying matrices:

- (AB)C = A(BC)
- k(AB) = (kA)B = A(kB), k is scalar (number)
- A(B $\pm$ C) = AB $\pm$ AC and (B $\pm$ C)A = BA $\pm$ CA
- AB ≠ BA
- OA = AO = O, O is zero matrix
- IA = AI = A, I is Identity matrix

MATRICES AND MATRIX OPERATIONS

TRANSPOSE OF THE MATRIX – It is a new matrix that we get when rows and columns of the matrix A change places, transpose of A is denoted by A^T. If matrix A has size m x n, then matrix A^T will have size n x m, because we have changed row and columns.

a) $A = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 4 & 6 \end{bmatrix}$ A^T =? b) $B = \begin{bmatrix} 7 & 2 & 4 \\ -2 & 3 & 5 \\ 1 & -8 & 9 \end{bmatrix}$ B^T = ?

To find A^T rows and columns of matrix A need to change places. We will put the first row [2 1 5] as a first column of A^T, and the second row [3 4 6] as second column of the A^T. Do the same for B^T.

a) $A^T = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 6 \end{bmatrix}$ b) $B^T = \begin{bmatrix} 7 & -2 & 1 \\ 2 & 3 & -8 \\ 4 & 5 & 9 \end{bmatrix}$

Note: If matrix is squared and $A = A^T$ we say that it is SYMMETRIC MATRIX.

$A = \begin{bmatrix} 5 & -2 & -1 \\ -2 & 4 & 7 \\ -1 & 7 & 6 \end{bmatrix}$ if we change places for rows and columns we will get $A^T = \begin{bmatrix} 5 & -2 & -1 \\ -2 & 4 & 7 \\ -1 & 7 & 6 \end{bmatrix}$

From here we can see $A = A^T$, thus this matrix is symmetric.

Rules for transpose (if the sizes of matrices are such that stated operations can be performed)**:**

- $(A^T)^T = A$
- $(kA)^T = kA^T$, k is scalar (number)
- $(A \pm B)^T = A^T \pm B^T$
- $(AB)^T = B^T A^T$

MINORS OF MATRIX – In this handout we will only cover minors for 3x3 matrices, but similarly it can be calculated for any squared matrix

For doing this we need to know determinate of the matrix 2x2.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

If A is a squared matrix, then the minor, denoted by M_{ij}, of element a_{ij} is the determinate of submatrix that remains after the *i*th row and *j*th column are deleted from A.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3x3} \quad det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

MATRICES AND MATRIX OPERATIONS

If we want M_{11}, since that is the minor that correspond to element a_{11}, we are going to cover row 1 and column 1, everything that is left we will write in the same order in our minor.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \qquad M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Now we are going to do the same thing for every minor.

$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$ $M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$ $M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$ $M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$

$M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$ $M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$ $M_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

COFACTORS - the number $\mathbf{C_{ij} = (-1)^{i+j} \times Mij}$ is the cofactor of element a_{ij}

ADJOINT OF THE MATRIX (ADJUGATE)

$$\mathbf{M} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = \begin{bmatrix} +M_{11} & -M_{12} & +M_{13} \\ -M_{21} & +M_{22} & -M_{23} \\ +M_{31} & -M_{32} & +M_{33} \end{bmatrix} = \begin{bmatrix} +\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ +\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{bmatrix}$$

$$\boldsymbol{adj}(\mathbf{A}) = \mathbf{M^T}$$

Note that signs (+ or -) are easy to remember where to put the right one. Start with + and then -, +, -, +, etc.

INVERSE OF MATRIX -If A is the square matrix and B is the same size of A. If matrix B can be find such that AB = BA = I, then A is said to be invertible (nonsingular, if $det(A) \neq 0$), and B is called an inverse of A. If there is no such matrix B, then A is not invertible (singular, if $det(A) = 0$).

Notation for the inverse of matrix A is A^{-1}. If B is inverse of A, then $B = A^{-1}$.

$$AB = BA = I \text{ or } AA^{-1} = A^{-1}A = I$$

Inverse of matrix A (for all squared matrices) can be found using this formula: $\mathbf{A^{-1}} = \frac{\mathbf{1}}{\boldsymbol{det(A)}} \boldsymbol{adj(A)}$.

<u>Inverse of 2x2 matrix</u> $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $A^{-1} = \frac{\mathbf{1}}{\boldsymbol{det(A)}} adj(A) = \frac{\mathbf{1}}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, if and only if $det(A) \neq 0$.

MATRICES AND MATRIX OPERATIONS

a) A = $\begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix}$, $A^{-1} = \frac{\mathbf{1}}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{\mathbf{1}}{6x2-1x5}\begin{bmatrix} 2 & -1 \\ -5 & 6 \end{bmatrix} = \frac{\mathbf{1}}{7}\begin{bmatrix} 2 & -1 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} \frac{\mathbf{2}}{\mathbf{7}} & -\frac{1}{7} \\ -\frac{\mathbf{5}}{\mathbf{7}} & \frac{\mathbf{6}}{\mathbf{7}} \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 5 & 0 \\ 0 & 3 & 2 \\ 1 & 0 & 2 \end{bmatrix}_{3x3}$, A^{-1} = ?

$M_{11} = \begin{vmatrix} 3 & 2 \\ 0 & 2 \end{vmatrix}$ = 3x2 – 0x2 = 6 $\quad M_{12} = \begin{vmatrix} 0 & 2 \\ 1 & 2 \end{vmatrix} = -2 \quad M_{13} = \begin{vmatrix} 0 & 3 \\ 1 & 0 \end{vmatrix}$ = -3

$M_{21} = \begin{vmatrix} 5 & 0 \\ 0 & 2 \end{vmatrix}$ = 10 $\quad M_{22} = \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} = 2 \quad M_{23} = \begin{vmatrix} 1 & 5 \\ 1 & 0 \end{vmatrix}$ = -5

$M_{31} = \begin{vmatrix} 5 & 0 \\ 3 & 2 \end{vmatrix}$ = 10 $\quad M_{32} = \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = 2 \quad M_{33} = \begin{vmatrix} 1 & 5 \\ 0 & 3 \end{vmatrix}$ = 3

$$M = \begin{bmatrix} +6 & -(-2) & +(-3) \\ -10 & +2 & -(-5) \\ +10 & -2 & +3 \end{bmatrix} = \begin{bmatrix} 6 & 2 & -3 \\ -10 & 2 & 5 \\ 10 & -2 & 3 \end{bmatrix} \quad adj(A) = M^T = \begin{bmatrix} 6 & -10 & 10 \\ 2 & 2 & -2 \\ -3 & 5 & 3 \end{bmatrix}$$

Help – Determinant of only 3x3 matrix can be find using Sarrus' rule. We write first two columns of the determinate to the right of the determinant (in that order). Then we are adding the products of the diagonals, going from the top to bottom (dashed lines), and subtract products of the diagonals going from the bottom to the top (solid lines).

$$det(A) = \begin{vmatrix} 1 & 5 & 0 \\ 0 & 3 & 2 \\ 1 & 0 & 2 \end{vmatrix} \begin{matrix} 1 & 5 \\ 0 & 3 \\ 1 & 0 \end{matrix} = 1x3x2 + 5x2x1 + 0x0x0 - 1x3x0 - 0x2x1 - 2x0x5 = 6 + 10 + 0 - 0 - 0 - 0 = 16$$

$$\text{Finally, } A^{-1} = \frac{1}{det(A)} adj(A) = \frac{1}{16}\begin{bmatrix} 6 & -10 & 10 \\ 2 & 2 & -2 \\ -3 & 5 & 3 \end{bmatrix} = \begin{bmatrix} \frac{6}{16} & \frac{-10}{16} & \frac{10}{16} \\ \frac{2}{16} & \frac{2}{16} & \frac{-2}{16} \\ -\frac{3}{16} & \frac{5}{16} & \frac{3}{16} \end{bmatrix} = \begin{bmatrix} \frac{3}{18} & \frac{-5}{8} & \frac{5}{8} \\ \frac{1}{8} & \frac{1}{8} & \frac{-1}{8} \\ -\frac{3}{16} & \frac{5}{16} & \frac{3}{16} \end{bmatrix}.$$

Rules for inverse (if A is invertible):

- $(A^{-1})^T = (A^T)^{-1}$
- $(A^{-1})^{-1} = A$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(kA)^{-1} = k^{-1}A^{-1} = \frac{1}{k}A^{-1}$, k is nonzero scalar
- $(A_1A_2 \ldots A_n)^{-1} = A_n^{-1} \ldots A_2^{-1}A_1^{-1}$

References: The following work were referred to during the creation of this handout: *Elementary Linear Algebra, Application Version, 11th Ed., Howard Anthon, Chriss Roress; and http://www.matematiranje.in.rs*