Method of Variation of Parameters (Higher Order)

For higher order differential equations, we can use the method of variation of parameters to get a general solution.

Steps for Solving Higher Order Nonhomogeneous Equations (Variation of Parameters):

1. Find the general solution to the corresponding homogeneous equation.
2. Find the particular solution to the nonhomogeneous equation by solving for $v_1(x)$, $v_2(x)$, …,$v_n(x)$, where the particular solution is given by $y_P(x) = v_1 y_1 + v_2 y_2 + ... + v_n y_n$ and the solutions of $v_1(x)$, $v_2(x)$, …,$v_n(x)$ are given by the following general formula:

$$v_n = \int \frac{g(x)W_n(x)}{W[y_1, y_2, ..., y_n](x)} dx$$

Note: Recall that $W[y_1(x),y_2(x),…,y_n(x)]$ is the Wronskian of $y_1(x)$, $y_2(x)$,…,$y_n(x)$ given by:

$$W[y_1, y_2, ..., y_n](x) = \begin{vmatrix} y_1(x) & y_2(x) & ... & y_n(x) \\ y_1'(x) & y_2'(x) & ... & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{n-1}(x) & y_2^{n-1}(x) & ... & y_n^{n-1}(x) \end{vmatrix}$$

Note: The $W_n(x)$ in the formula is given by

$$W_n(x) = (-1)^{n-k} W[y_1, ... y_{k-1}, y_{k+1}, ..., y_n](x),$$

where n is the number of fundamental solutions u have and k = 1, 2, …, n.

3. Write the general solution to the nonhomogeneous differential equation by adding the homogeneous solution to the particular solution to get the form $y = y_H + y_P$.

Example:

Find the general solution to the differential equation $y''' + y' = \tan x$.

1. Find a general solution the associated homogeneous equation.

$$y''' + y' = 0$$
$$r^3 + r = 0$$
$$r(r^2 + 1) = 0$$
$$r = 0, r = \pm i$$

--Thus, $y_H = c_1 + c_2 \cos x + c_3 \sin x$.

2. Find the particular solution to the nonhomogeneous equation.

--We know that the fundamental solution is $\{y_1,y_2,y_3\} = \{1, \cos x, \sin x\}$. *Calculate the Wronskian of the fundamental set.*

$$W[1, \cos x, \sin x] = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix}$$

--Using cofactor expansion using the first column yields:

$$\begin{vmatrix} \boxed{1} & \cos x & \sin x \\ \boxed{0} & -\sin x & \cos x \\ \boxed{0} & -\cos x & -\sin x \end{vmatrix} = 1\begin{vmatrix} -\sin x & \cos x \\ -\cos x & -\sin x \end{vmatrix} - 0\begin{vmatrix} \cos x & \sin x \\ -\cos x & -\sin x \end{vmatrix} - 0\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$= \sin^2 x + \cos^2 x = 1$$

--Finding the other Wronskians gives the following:

$$W_1(x) = (-1)^{3-1}W[\cos x, \sin x](x) = (-1)^2\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$W_2(x) = (-1)^{3-2}W[1, \sin x](x) = (-1)\begin{vmatrix} 1 & \sin x \\ 0 & \cos x \end{vmatrix} = (-1)(-\sin x) = \sin x$$

$$W_3(x) = (-1)^{3-3}W[1, \cos x](x) = (-1)^3\begin{vmatrix} 1 & \cos x \\ 0 & -\sin x \end{vmatrix} = \cos x$$

--Find $v_1(x)$, $v_2(x)$, and $v_3(x)$ by plugging into the formula above metioned in step 2.

$$v_1(x) = \int \frac{g(x)W_1(x)}{W[y_1(x), y_2(x), y_3(x)]}dx = \int \frac{(\tan x)(1)}{1}dx = \int \tan x + C = -\ln|\cos x| + C$$

$$v_3(x) = \int \frac{g(x)W_3(x)}{W[y_1(x), y_2(x), y_3(x)]}dx = \int \frac{(\tan x)(\cos x)}{1}dx = \int\left(\frac{\sin x}{\cos x}\cdot\cos x\right)dx = \int \sin x dx = -\cos x + C$$

$$v_2(x) = \int \frac{g(x)W_2(x)}{W[y_1(x), y_2(x), y_3(x)]}dx = \int \frac{(\tan x)(\sin x)}{1}dx = \int(\tan x \sin x)dx = \int\left(\frac{\sin x}{\cos x}\cdot\sin x\right)dx = \int\left(\frac{\sin^2 x}{\cos x}\right)dx = \int\left(\frac{1-\cos^2 x}{\cos x}\right)dx = \int\left(\frac{1}{\cos x} - \frac{\cos^2 x}{\cos x}\right)dx = \int(\sec x - \cos x)dx = \ln|\sec x + \tan x| - \sin x + C$$

--Thus

$$y_P(x) = (1)(1) + (\cos x)(\ln|\sec x + \tan x| - \sin x + C_2) + (\sin x)(-\cos x + C_3)$$
$$= 1 + (\cos x)(\ln|\sec x + \tan x|) - \sin x\cos x - \sin x \cos x$$
$$= 1 + (\cos x)(\ln|\sec x + \tan x|) - 2\sin x\cos x$$
$$= 1 + (\cos x)(\ln|\sec x + \tan x|) - \sin 2x$$

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Note: We took the constants to be zero to make the solution more feasible.

3. Write the general solution by adding together the homogeneous solution and the particular solution.

Hence, the general solution is given by

$$y = c_1 + c_2 \cos x + c_3 \sin x + (\cos x)(\ln|\sec x + \tan x|) - \sin 2x \,.$$

(Remove bottom row and *i* th column.)

If $n = 3$, then

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}.$$

Therefore

$$W_1 = \begin{vmatrix} y_2 & y_3 \\ y_2' & y_3' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & y_3 \\ y_1' & y_3' \end{vmatrix}, \quad W_3 = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix},$$