

## ***Intervals of Validity***

I've called this section Intervals of Validity because all of the examples will involve them. However, there is a lot more to this section. We will see a couple of theorems that will tell us when we can solve a differential equation. We will also see some of the differences between linear and nonlinear differential equations.

First let's take a look at a theorem about linear first order differential equations. This is a very important theorem although we're not going to really use it for its most important aspect.

### ***Theorem 1***

Consider the following IVP.

$$y' + p(t)y = g(t) \qquad y(t_0) = y_0$$

If  $p(t)$  and  $g(t)$  are continuous functions on an open interval  $\alpha < t < \beta$  and the interval contains  $t_0$ , then there is a unique solution to the IVP on that interval.

So, just what does this theorem tell us? First, it tells us that for nice enough linear first order differential equations solutions are guaranteed to exist and more importantly the solution will be unique. We may not be able to find the solution, but do know that it exists and that there will only be one of them. This is the very important aspect of this theorem. Knowing that a differential equation has a unique solution is sometimes more important than actually having the solution itself!

Next, if the interval in the theorem is the largest possible interval on which  $p(t)$  and  $g(t)$  are continuous then the interval is the interval of validity for the solution. This means, that for linear first order differential equations, we won't need to actually solve the differential equation in order to find the interval of validity. Notice as well that the interval of validity will depend only partially on the initial condition. The interval must contain  $t_0$ , but the value of  $y_0$ , has no effect on the interval of validity.

Let's take a look at an example.

***Example 1*** Without solving, determine the interval of validity for the following initial value problem.

$$(t^2 - 9)y' + 2y = \ln|20 - 4t| \qquad y(4) = -3$$

### ***Solution***

First, in order to use the theorem to find the interval of validity we must write the differential equation in the proper form given in the theorem. So we will need to divide out by the coefficient of the derivative.

$$y' + \frac{2}{t^2 - 9}y = \frac{\ln|20 - 4t|}{t^2 - 9}$$

Next, we need to identify where the two functions are not continuous. This will allow us to find all possible intervals of validity for the differential equation. So,  $p(t)$  will be discontinuous at  $t = \pm 3$  since these points will give a division by zero. Likewise,  $g(t)$  will also be discontinuous at  $t = \pm 3$  as well as  $t = 5$  since at this point we will have the natural logarithm of zero. Note that in

this case we won't have to worry about natural log of negative numbers because of the absolute values.

Now, with these points in hand we can break up the real number line into four intervals where both  $p(t)$  and  $g(t)$  will be continuous. These four intervals are,

$$-\infty < t < -3 \qquad -3 < t < 3 \qquad 3 < t < 5 \qquad 5 < t < \infty$$

The endpoints of each of the intervals are points where at least one of the two functions is discontinuous. This will guarantee that both functions are continuous everywhere in each interval.

Finally, let's identify the actual interval of validity for the initial value problem. The actual interval of validity is the interval that will contain  $t_0 = 4$ . So, the interval of validity for the initial value problem is.

$$3 < t < 5$$

In this last example we need to be careful to not jump to the conclusion that the other three intervals cannot be intervals of validity. By changing the initial condition, in particular the value of  $t_0$ , we can make any of the four intervals the interval of validity.

The first theorem required a linear differential equation. There is a similar theorem for non-linear first order differential equations. This theorem is not as useful for finding intervals of validity as the first theorem was so we won't be doing all that much with it.

Here is the theorem.

**Theorem 2**

Consider the following IVP.

$$y' = f(t, y) \qquad y(t_0) = y_0$$

If  $f(t, y)$  and  $\frac{\partial f}{\partial y}$  are continuous functions in some rectangle  $\alpha < t < \beta$ ,  $\gamma < y < \delta$  containing the point  $(t_0, y_0)$  then there is a unique solution to the IVP in some interval  $t_0 - h < t < t_0 + h$  that is contained in  $\alpha < t < \beta$ .

That's it. Unlike the first theorem, this one cannot really be used to find an interval of validity. So, we will know that a unique solution exists if the conditions of the theorem are met, but we will actually need the solution in order to determine its interval of validity. Note as well that for non-linear differential equations it appears that the value of  $y_0$  may affect the interval of validity.

Here is an example of the problems that can arise when the conditions of this theorem aren't met.

**Example 2** Determine all possible solutions to the following IVP.

$$y' = y^{\frac{1}{3}} \qquad y(0) = 0$$

**Solution**

First, notice that this differential equation does NOT satisfy the conditions of the theorem.

$$f(y) = y^{\frac{1}{3}} \qquad \frac{df}{dy} = \frac{1}{3y^{\frac{2}{3}}}$$

So, the function is continuous on any interval, but the derivative is not continuous at  $y = 0$  and so will not be continuous at any interval containing  $y = 0$ . In order to use the theorem both must be continuous on an interval that contains  $y_0 = 0$  and this is a problem for us since we do have  $y_0 = 0$ .

Now, let's actually work the problem. This differential equation is separable and is fairly simple to solve.

$$\int y^{-\frac{1}{3}} dy = \int dt$$

$$\frac{3}{2} y^{\frac{2}{3}} = t + c$$

Applying the initial condition gives  $c = 0$  and so the solution is.

$$\frac{3}{2} y^{\frac{2}{3}} = t$$

$$y^{\frac{2}{3}} = \frac{2}{3} t$$

$$y^2 = \left( \frac{2}{3} t \right)^3$$

$$y(t) = \pm \left( \frac{2}{3} t \right)^{\frac{3}{2}}$$

So, we've got two possible solutions here, both of which satisfy the differential equation and the initial condition. There is also a third solution to the IVP.  $y(t) = 0$  is also a solution to the differential equation and satisfies the initial condition.

In this last example we had a very simple IVP and it only violated one of the conditions of the theorem, yet it had three different solutions. All the examples we've worked in the previous sections satisfied the conditions of this theorem and had a single unique solution to the IVP. This example is a useful reminder of the fact that, in the field of differential equations, things don't always behave nicely. It's easy to forget this as most of the problems that are worked in a differential equations class are nice and behave in a nice, predictable manner.

Let's work one final example that will illustrate one of the differences between linear and non-linear differential equations.

**Example 3** Determine the interval of validity for the initial value problem below and give its dependence on the value of  $y_0$ .

$$y' = y^2 \quad y(0) = y_0$$

**Solution**

Before proceeding in this problem, we should note that the differential equation is non-linear and meets both conditions of the Theorem 2 and so there will be a unique solution to the IVP for each possible value of  $y_0$ .

Also, note that the problem asks for any dependence of the interval of validity on the value of  $y_0$ . This immediately illustrates a difference between linear and non-linear differential equations.

Intervals of validity for linear differential equations do not depend on the value of  $y_0$ . Intervals of validity for non-linear differential can depend on the value of  $y_0$  as we pointed out after the second theorem.

So, let's solve the IVP and get some intervals of validity.

First note that if  $y_0 = 0$  then  $y(t) = 0$  is the solution and this has an interval of validity of

$$-\infty < t < \infty$$

So for the rest of the problem let's assume that  $y_0 \neq 0$ . Now, the differential equation is separable so let's solve it and get a general solution.

$$\int y^{-2} dy = \int dt$$

$$-\frac{1}{y} = t + c$$

Applying the initial condition gives

$$c = -\frac{1}{y_0}$$

The solution is then.

$$-\frac{1}{y} = t - \frac{1}{y_0}$$

$$y(t) = \frac{1}{\frac{1}{y_0} - t}$$

$$y(t) = \frac{y_0}{1 - y_0 t}$$

Now that we have a solution to the initial value problem we can start finding intervals of validity. From the solution we can see that the only problem that we'll have is division by zero at

$$t = \frac{1}{y_0}$$

This leads to two possible intervals of validity.

$$-\infty < t < \frac{1}{y_0}$$

$$\frac{1}{y_0} < t < \infty$$

That actual interval of validity will be the interval that contains  $t_0 = 0$ . This however, depends on the value of  $y_0$ . If  $y_0 < 0$  then  $\frac{1}{y_0} < 0$  and so the second interval will contain  $t_0 = 0$ . Likewise if  $y_0 > 0$  then  $\frac{1}{y_0} > 0$  and in this case the first interval will contain  $t_0 = 0$ .

This leads to the following possible intervals of validity, depending on the value of  $y_0$ .

If $y_0 > 0$	$-\infty < t < \frac{1}{y_0}$ is the interval of validity.
If $y_0 = 0$	$-\infty < t < \infty$ is the interval of validity.
If $y_0 < 0$	$\frac{1}{y_0} < t < \infty$ is the interval of validity.

On a side note, notice that the solution, in its final form, will also work if  $y_0 = 0$ .

So what did this example show us about the difference between linear and non-linear differential equations?

First, as pointed out in the solution to the example, intervals of validity for non-linear differential equations can depend on the value of  $y_0$ , whereas intervals of validity for linear differential equations don't.

Second, intervals of validity for linear differential equations can be found from the differential equation with no knowledge of the solution. This is definitely not the case with non-linear differential equations. It would be very difficult to see how any of these intervals in the last example could be found from the differential equation. Knowledge of the solution was required in order for us to find the interval of validity.