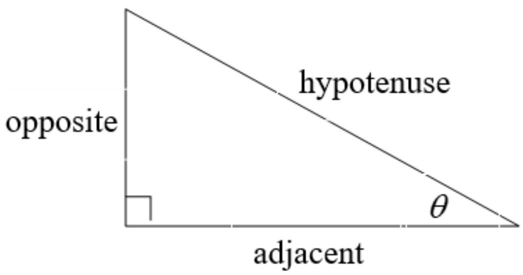
Definition of the Trig Functions

Right triangle definition

For this definition we assume that $0 < \theta < \frac{\pi}{2}$ or $0° < \theta < 90°$.



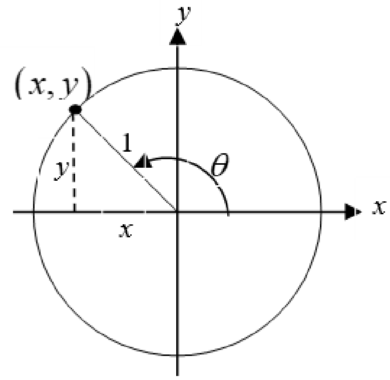
$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$ $\csc(\theta) = \frac{\text{hypotenuse}}{\text{opposite}}$

$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$ $\sec(\theta) = \frac{\text{hypotenuse}}{\text{adjacent}}$

$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$ $\cot(\theta) = \frac{\text{adjacent}}{\text{opposite}}$

Unit Circle Definition

For this definition θ is any angle.



$\sin(\theta) = \frac{y}{1} = y$ $\csc(\theta) = \frac{1}{y}$

$\cos(\theta) = \frac{x}{1} = x$ $\sec(\theta) = \frac{1}{x}$

$\tan(\theta) = \frac{y}{x}$ $\cot(\theta) = \frac{x}{y}$

Facts and Properties

Domain

The domain is all the values of θ that can be plugged into the function.

$\sin(\theta)$, θ can be any angle

$\cos(\theta)$, θ can be any angle

$\tan(\theta)$, $\theta \neq \left(n + \frac{1}{2}\right)\pi, \; n = 0, \pm 1, \pm 2, \ldots$

$\csc(\theta)$, $\theta \neq n\pi, \; n = 0, \pm 1, \pm 2, \ldots$

$\sec(\theta)$, $\theta \neq \left(n + \frac{1}{2}\right)\pi, \; n = 0, \pm 1, \pm 2, \ldots$

$\cot(\theta)$, $\theta \neq n\pi, \; n = 0, \pm 1, \pm 2, \ldots$

Period

The period of a function is the number, T, such that $f(\theta + T) = f(\theta)$. So, if ω is a fixed number and θ is any angle we have the following periods.

$$\sin(\omega\,\theta) \rightarrow T = \frac{2\pi}{\omega}$$

$$\cos(\omega\,\theta) \rightarrow T = \frac{2\pi}{\omega}$$

$$\tan(\omega\,\theta) \rightarrow T = \frac{\pi}{\omega}$$

$$\csc(\omega\,\theta) \rightarrow T = \frac{2\pi}{\omega}$$

$$\sec(\omega\,\theta) \rightarrow T = \frac{2\pi}{\omega}$$

$$\cot(\omega\,\theta) \rightarrow T = \frac{\pi}{\omega}$$

Range

The range is all possible values to get out of the function.

$-1 \leq \sin(\theta) \leq 1$ $-1 \leq \cos(\theta) \leq 1$

$-\infty < \tan(\theta) < \infty$ $-\infty < \cot(\theta) < \infty$

$\sec(\theta) \geq 1$ and $\sec(\theta) \leq -1$ $\csc(\theta) \geq 1$ and $\csc(\theta) \leq -1$

Formulas and Identities

Tangent and Cotangent Identities

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \qquad \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$$

Reciprocal Identities

$$\csc(\theta) = \frac{1}{\sin(\theta)} \qquad \sin(\theta) = \frac{1}{\csc(\theta)}$$

$$\sec(\theta) = \frac{1}{\cos(\theta)} \qquad \cos(\theta) = \frac{1}{\sec(\theta)}$$

$$\cot(\theta) = \frac{1}{\tan(\theta)} \qquad \tan(\theta) = \frac{1}{\cot(\theta)}$$

Pythagorean Identities

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

$$\tan^2(\theta) + 1 = \sec^2(\theta)$$

$$1 + \cot^2(\theta) = \csc^2(\theta)$$

Even/Odd Formulas

$$\sin(-\theta) = -\sin(\theta) \qquad \csc(-\theta) = -\csc(\theta)$$

$$\cos(-\theta) = \cos(\theta) \qquad \sec(-\theta) = \sec(\theta)$$

$$\tan(-\theta) = -\tan(\theta) \qquad \cot(-\theta) = -\cot(\theta)$$

Periodic Formulas

If n is an integer then,

$$\sin(\theta + 2\pi n) = \sin(\theta) \qquad \csc(\theta + 2\pi n) = \csc(\theta)$$

$$\cos(\theta + 2\pi n) = \cos(\theta) \qquad \sec(\theta + 2\pi n) = \sec(\theta)$$

$$\tan(\theta + \pi n) = \tan(\theta) \qquad \cot(\theta + \pi n) = \cot(\theta)$$

Degrees to Radians Formulas

If x is an angle in degrees and t is an angle in radians then

$$\frac{\pi}{180} = \frac{t}{x} \quad \Rightarrow \quad t = \frac{\pi x}{180} \quad \text{and} \quad x = \frac{180t}{\pi}$$

Double Angle Formulas

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$$\begin{aligned} \cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) \\ &= 2\cos^2(\theta) - 1 \\ &= 1 - 2\sin^2(\theta) \end{aligned}$$

$$\tan(2\theta) = \frac{2\tan(\theta)}{1 - \tan^2(\theta)}$$

Half Angle Formulas

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1 - \cos(\theta)}{2}}$$

$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1 + \cos(\theta)}{2}}$$

$$\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1 - \cos(\theta)}{1 + \cos(\theta)}}$$

Half Angle Formulas (alternate form)

$$\sin^2(\theta) = \tfrac{1}{2}\left(1 - \cos(2\theta)\right) \qquad \tan^2(\theta) = \frac{1 - \cos(2\theta)}{1 + \cos(2\theta)}$$

$$\cos^2(\theta) = \tfrac{1}{2}\left(1 + \cos(2\theta)\right)$$

Sum and Difference Formulas

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$$

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$$

$$\tan(\alpha \pm \beta) = \frac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha)\tan(\beta)}$$

Product to Sum Formulas

$$\sin(\alpha)\sin(\beta) = \tfrac{1}{2}\left[\cos(\alpha - \beta) - \cos(\alpha + \beta)\right]$$

$$\cos(\alpha)\cos(\beta) = \tfrac{1}{2}\left[\cos(\alpha - \beta) + \cos(\alpha + \beta)\right]$$

$$\sin(\alpha)\cos(\beta) = \tfrac{1}{2}\left[\sin(\alpha + \beta) + \sin(\alpha - \beta)\right]$$

$$\cos(\alpha)\sin(\beta) = \tfrac{1}{2}\left[\sin(\alpha + \beta) - \sin(\alpha - \beta)\right]$$

Sum to Product Formulas

$$\sin(\alpha) + \sin(\beta) = 2\sin\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right)$$

$$\sin(\alpha) - \sin(\beta) = 2\cos\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right)$$

$$\cos(\alpha) + \cos(\beta) = 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right)$$

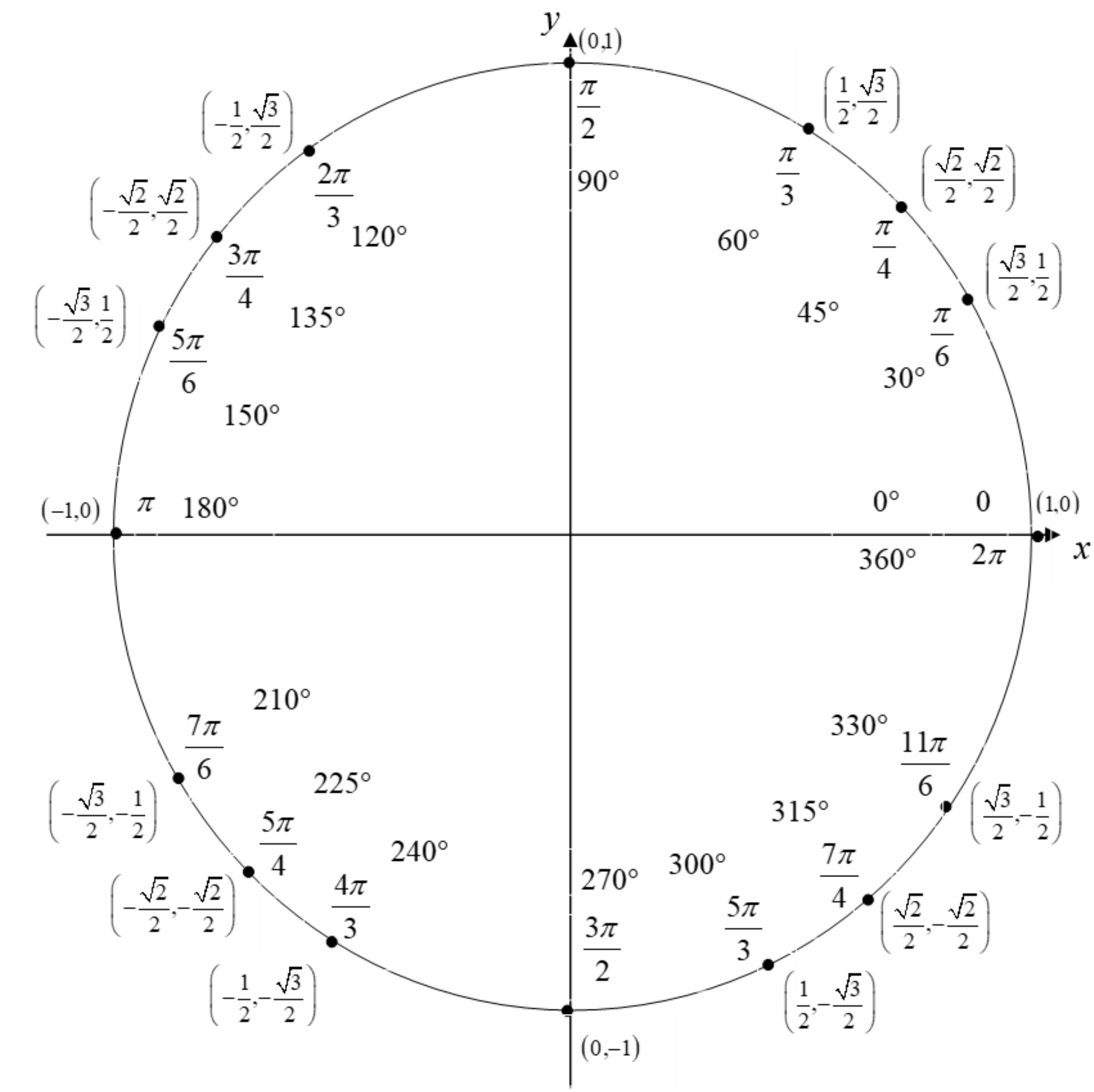
$$\cos(\alpha) - \cos(\beta) = -2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right)$$

Cofunction Formulas

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos(\theta) \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$$

$$\csc\left(\frac{\pi}{2} - \theta\right) = \sec(\theta) \qquad \sec\left(\frac{\pi}{2} - \theta\right) = \csc(\theta)$$

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot(\theta) \qquad \cot\left(\frac{\pi}{2} - \theta\right) = \tan(\theta)$$



For any ordered pair on the unit circle (x, y) : cos$(\theta) = x$ and sin$(\theta) = y$

Example

$$\cos\left(\frac{5\pi}{3}\right) = \frac{1}{2} \qquad \sin\left(\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

Inverse Trig Functions

Definition

$y = \sin^{-1}(x)$ is equivalent to $x = \sin(y)$

$y = \cos^{-1}(x)$ is equivalent to $x = \cos(y)$

$y = \tan^{-1}(x)$ is equivalent to $x = \tan(y)$

Domain and Range

Function	Domain	Range
$y = \sin^{-1}(x)$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \cos^{-1}(x)$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \tan^{-1}(x)$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$

Inverse Properties

$\cos\left(\cos^{-1}(x)\right) = x \qquad \cos^{-1}\left(\cos(\theta)\right) = \theta$

$\sin\left(\sin^{-1}(x)\right) = x \qquad \sin^{-1}\left(\sin(\theta)\right) = \theta$

$\tan\left(\tan^{-1}(x)\right) = x \qquad \tan^{-1}\left(\tan(\theta)\right) = \theta$

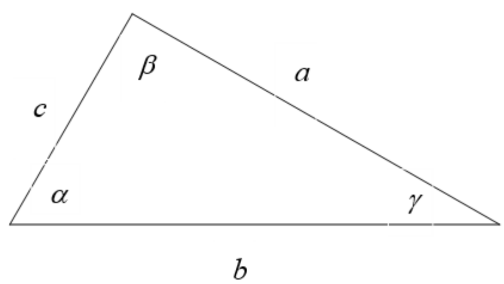
Alternate Notation

$\sin^{-1}(x) = \arcsin(x)$

$\cos^{-1}(x) = \arccos(x)$

$\tan^{-1}(x) = \arctan(x)$

Law of Sines, Cosines and Tangents



Law of Sines

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}$$

Law of Cosines

$$a^2 = b^2 + c^2 - 2bc\cos(\alpha)$$

$$b^2 = a^2 + c^2 - 2ac\cos(\beta)$$

$$c^2 = a^2 + b^2 - 2ab\cos(\gamma)$$

Mollweide's Formula

$$\frac{a+b}{c} = \frac{\cos\left(\frac{1}{2}(\alpha-\beta)\right)}{\sin\left(\frac{1}{2}\gamma\right)}$$

Law of Tangents

$$\frac{a-b}{a+b} = \frac{\tan\left(\frac{1}{2}(\alpha-\beta)\right)}{\tan\left(\frac{1}{2}(\alpha+\beta)\right)}$$

$$\frac{b-c}{b+c} = \frac{\tan\left(\frac{1}{2}(\beta-\gamma)\right)}{\tan\left(\frac{1}{2}(\beta+\gamma)\right)}$$

$$\frac{a-c}{a+c} = \frac{\tan\left(\frac{1}{2}(\alpha-\gamma)\right)}{\tan\left(\frac{1}{2}(\alpha+\gamma)\right)}$$

Odd and even properties

$$\cos(-x) = \cos(x) \qquad \sin(-x) = -\sin(x) \qquad \tan(-x) = -\tan(x)$$

Unit circle properties

$$\begin{aligned} \cos(\pi - x) &= -\cos(x) & \sin(\pi - x) &= \sin(x) & \tan(\pi - x) &= -\tan(x) \\ \cos(\pi + x) &= -\cos(x) & \sin(\pi + x) &= -\sin(x) & \tan(\pi + x) &= \tan(x) \\ \cos(2\pi - x) &= \cos(x) & \sin(2\pi - x) &= -\sin(x) & \tan(2\pi - x) &= -\tan(x) \\ \cos(2\pi + x) &= \cos(x) & \sin(2\pi + x) &= \sin(x) & \tan(2\pi + x) &= \tan(x) \end{aligned}$$

Right-angled triangle properties

$$\cos\left(\tfrac{\pi}{2} - x\right) = \sin(x) \qquad \sin\left(\tfrac{\pi}{2} - x\right) = \cos(x)$$

Shifting by $\frac{\pi}{2}$

$$\begin{aligned} \cos(x) &= \cos(x) & \cos(x) &= \cos(x) & \cos(-x) &= \cos(x) \\ \cos(x + \tfrac{\pi}{2}) &= -\sin(x) & \cos(x - \tfrac{\pi}{2}) &= \sin(x) & \cos(\tfrac{\pi}{2} - x) &= \sin(x) \\ \cos(x + \pi) &= -\cos(x) & \cos(x - \pi) &= -\cos(x) & \cos(\pi - x) &= -\cos(x) \\ \cos(x + \tfrac{3\pi}{2}) &= \sin(x) & \cos(x - \tfrac{3\pi}{2}) &= -\sin(x) & \cos(\tfrac{3\pi}{2} - x) &= -\sin(x) \\ \cos(x + 2\pi) &= \cos(x) & \cos(x - 2\pi) &= \cos(x) & \cos(2\pi - x) &= \cos(x) \end{aligned}$$

$$\begin{aligned} \sin(x) &= \sin(x) & \sin(x) &= \sin(x) & \sin(-x) &= -\sin(x) \\ \sin(x + \tfrac{\pi}{2}) &= \cos(x) & \sin(x - \tfrac{\pi}{2}) &= -\cos(x) & \sin(\tfrac{\pi}{2} - x) &= \cos(x) \\ \sin(x + \pi) &= -\sin(x) & \sin(x - \pi) &= -\sin(x) & \sin(\pi - x) &= \sin(x) \\ \sin(x + \tfrac{3\pi}{2}) &= -\cos(x) & \sin(x - \tfrac{3\pi}{2}) &= \cos(x) & \sin(\tfrac{3\pi}{2} - x) &= -\cos(x) \\ \sin(x + 2\pi) &= \sin(x) & \sin(x - 2\pi) &= \sin(x) & \sin(2\pi - x) &= -\sin(x) \end{aligned}$$

$$\begin{aligned} \tan(x) &= \tan(x) & \tan(x) &= \tan(x) & \tan(-x) &= -\tan(x) \\ \tan(x + \tfrac{\pi}{2}) &= -\cot(x) & \tan(x - \tfrac{\pi}{2}) &= -\cot(x) & \tan(\tfrac{\pi}{2} - x) &= \cot(x) \\ \tan(x + \pi) &= \tan(x) & \tan(x - \pi) &= \tan(x) & \tan(\pi - x) &= -\tan(x) \\ \tan(x + \tfrac{3\pi}{2}) &= -\cot(x) & \tan(x - \tfrac{3\pi}{2}) &= -\cot(x) & \tan(\tfrac{3\pi}{2} - x) &= \cot(x) \\ \tan(x + 2\pi) &= \tan(x) & \tan(x - 2\pi) &= \tan(x) & \tan(2\pi - x) &= -\tan(x) \end{aligned}$$

Higher powers of sine and cosine

$$\cos^3 x = \cos x(\cos^2 x) = \cos x\left(\frac{1+\cos 2x}{2}\right)$$

$$= \frac{1}{2}\cos x + \frac{1}{2}(\cos x \cos 2x)$$

$$= \frac{1}{2}\cos x + \frac{1}{4}(2\cos x \cos 2x)$$

$$= \frac{1}{2}\cos x + \frac{1}{4}(\cos 3x + \cos(-x)) \because (\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2$$

$$= \frac{1}{2}\cos x + \frac{1}{4}(\cos 3x + \cos(x))$$

$$= \frac{3}{4}\cos x + \frac{1}{4}(\cos 3x)$$

$$= \frac{1}{4}(3\cos x + \cos 3x)$$

$$\sin^{2n} x = \frac{1}{2^{2n}}\binom{2n}{n} + \frac{(-1)^n}{2^{2n-1}}\sum_{k=0}^{n-1}(-1)^k\binom{2n}{k}\cos[2(n-k)x]$$

$$\sin^{2n+1} x = \frac{(-1)^n}{4^n}\sum_{k=0}^{n}(-1)^k\binom{2n+1}{k}\sin[(2n+1-2k)x]$$

$$\cos^{2n} x = \frac{1}{2^{2n}}\binom{2n}{n} + \frac{1}{2^{2n-1}}\sum_{k=0}^{n-1}\binom{2n}{k}\cos[2(n-k)x]$$

$$\cos^{2n+1} x = \frac{1}{4^n}\sum_{k=0}^{n}\binom{2n+1}{k}\cos[(2n+1-2k)x],$$

$$\sin^3 x = \frac{1}{4}[3\sin x - \sin(3x)]$$

$$\sin^4 x = \frac{1}{8}[3 - 4\cos(2x) + \cos(4x)]$$

$$\cos^3 x = \frac{1}{4}[3\cos x + \cos(3x)]$$

$$\cos^4 x = \frac{1}{8}[3 + 4\cos(2x) + \cos(4x)]$$