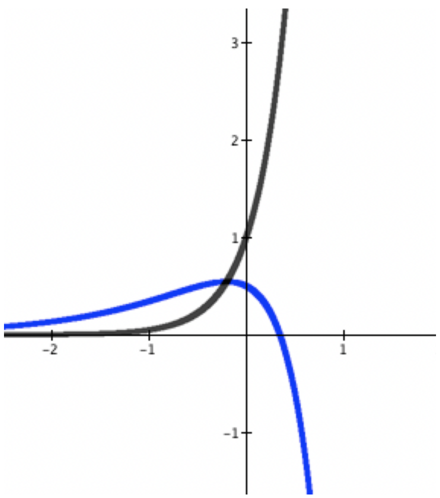
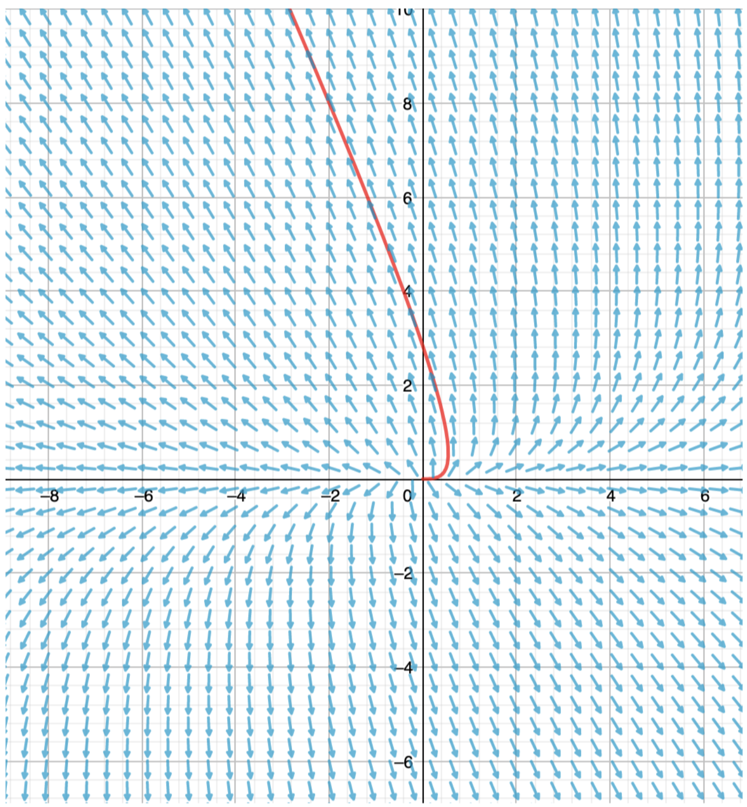
Systems of Ordinary Linear Differential Equations

Solve

$$\frac{dx}{dt} = x - y$$
$$\frac{dy}{dt} = 3y$$





$$x(t) = -\frac{1}{2}c_1 e^{3t} + c_2 e^{t}$$
$$y(t) = c_1 e^{3t}$$

Systems of Ordinary Linear Differential Equations

Using Elimination

Solve

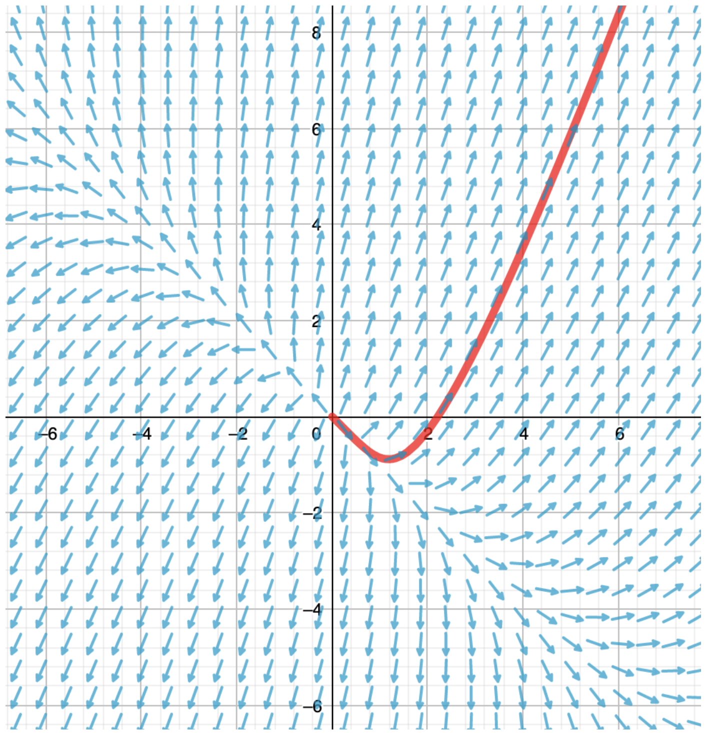
$$x' = 2x + y$$
$$y' = 3x + 4y$$

eliminate y so $y = x' - 2x$

$$(x' - 2x)' = 3x + 4(x' - 2x)$$

$$\boxed{x'' - 6x' + 5x = 0}$$

$$x = c_1 e^{t} + c_2 e^{5t}$$
$$y = -c_1 e^{t} + 3c_2 e^{5t}$$



Systems of Ordinary Linear Differential Equations

Using Laplace Transforms

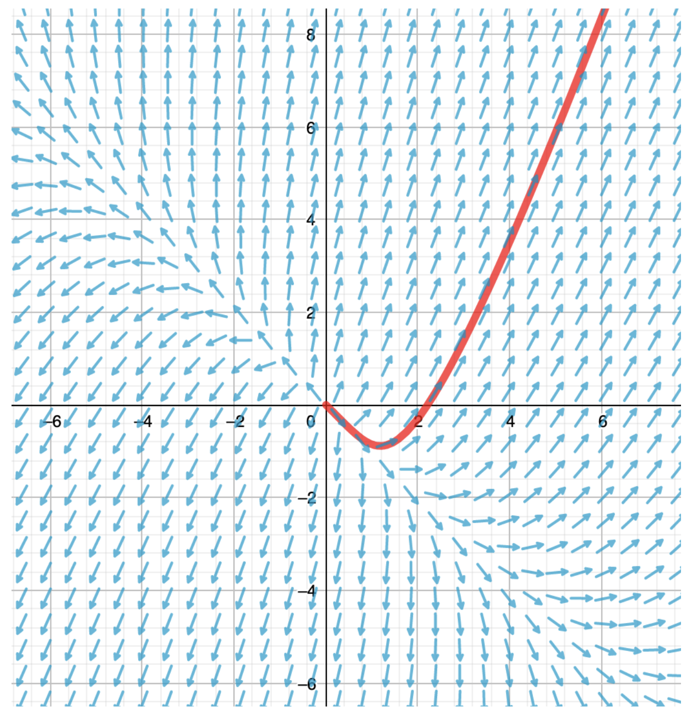
Solve

$$\frac{dx}{dt} = 2x + y; \quad x(0) = 1$$

$$\frac{dy}{dt} = 3x + 4y; \quad y(0) = 0$$

$$sX(s) - 1 = 2X(s) + Y(s) \quad (*)$$

$$sY(s) = 3X(s) + 4Y(s)$$



$$X(s) = \frac{s-4}{(s-1)(s-5)} = \frac{\frac{3}{4}}{(s-1)} + \frac{\frac{1}{4}}{(s-5)}$$

$$Y(s) = \frac{3}{(s-1)(s-5)} = \frac{-\frac{3}{4}}{(s-1)} + \frac{\frac{3}{4}}{(s-5)}$$

We can now use substition in (*) above to find that

$$x(t) = \frac{3}{4}e^{t} + \frac{1}{4}e^{5t}$$

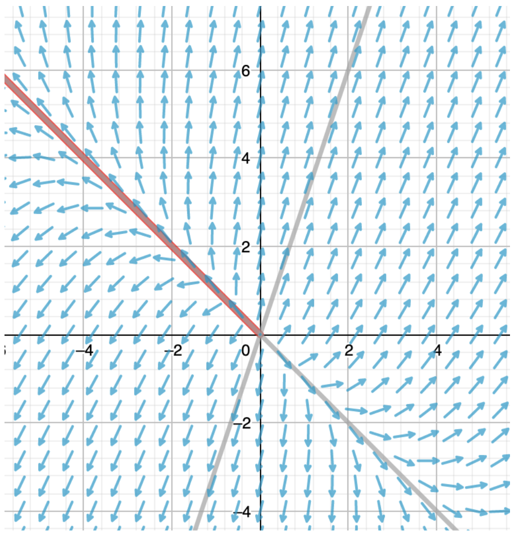
$$y(t) = -\frac{3}{4}e^{t} + \frac{3}{4}e^{5t}$$

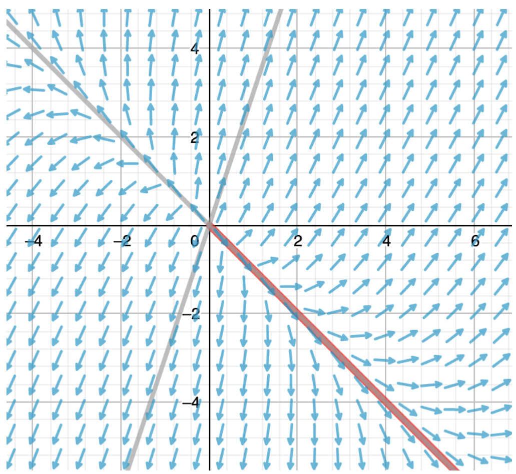
Systems of Ordinary Linear Differential Equations

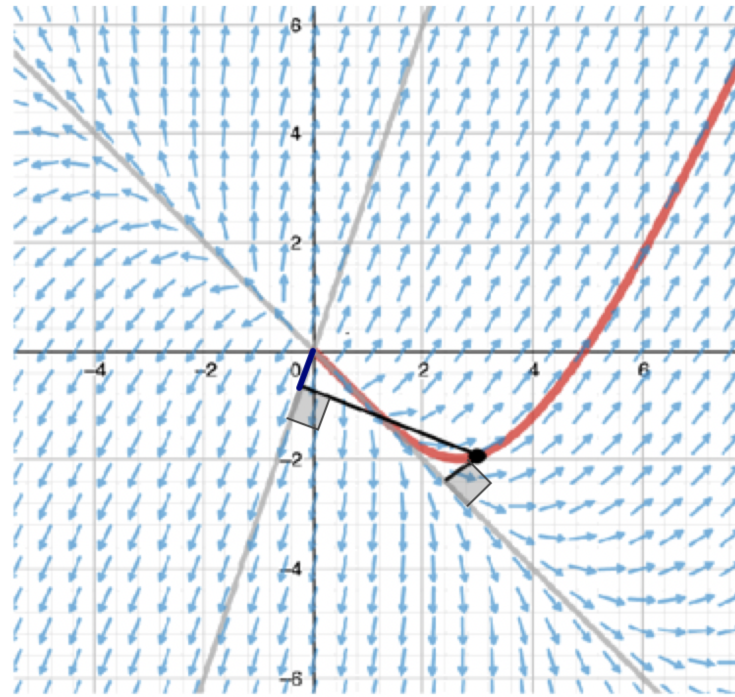
Eigenvalues and Eigenvectors

Solve

$$x' = 2x + y$$
$$y' = 3x + 4y$$

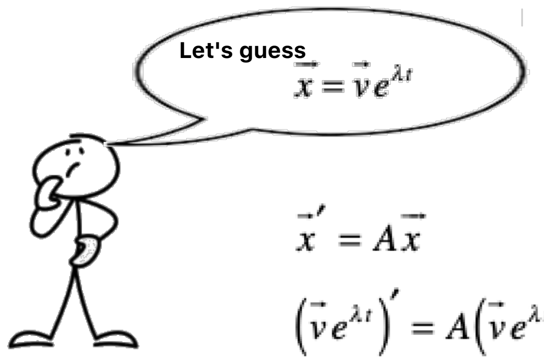






Systems of Ordinary Linear Differential Equations

Matrix Solutions



$$\vec{x}\,' = A\vec{x}$$

$$\left(\vec{v}e^{\lambda t}\right)' = A\left(\vec{v}e^{\lambda t}\right)$$

$$\lambda\vec{v}e^{\lambda t} = A\vec{v}e^{\lambda t}$$

$$\lambda\vec{v} = A\vec{v}$$

$$\vec{x}\,' = A\vec{x}$$

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \lambda\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \vec{\mathbf{0}}$$

$$(A - \lambda \boldsymbol{I})\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \vec{\mathbf{0}}$$

so $\vec{v}e^{\lambda t}$ is a solution if $\lambda\vec{v} = A\vec{v}$

λ is an **eigenvalue** and

$\vec{v}$ is its associated **eigenvector**

$$\boxed{\det(A - \lambda \boldsymbol{I}) = 0}$$

$$A\vec{x} = \vec{b}$$

$$A^{-1}A\vec{x} = A^{-1}\vec{b}$$

$$\vec{x} = A^{-1}\vec{b}$$

A^{-1} exists *iff* $\det(A) = 0$

$$\boxed{\lambda^2 - (\text{trace } A)\lambda + \det A = 0}$$

Systems of Ordinary Linear Differential Equations

Matrix Solutions

Solve

$$x' = 2x + y$$
$$y' = 3x + 4y$$

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} 2-\lambda & 1 \\ 3 & 4-\lambda \end{pmatrix}$$

$$\det(A - \lambda I) = \lambda^2 - 6\lambda + 5$$

$$\lambda^2 - 6\lambda + 5 = 0$$
$$\Rightarrow \lambda_1 = 1,\ \lambda_2 = 5$$

$\lambda_1 = 1$:

$$A - \lambda_1 I = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

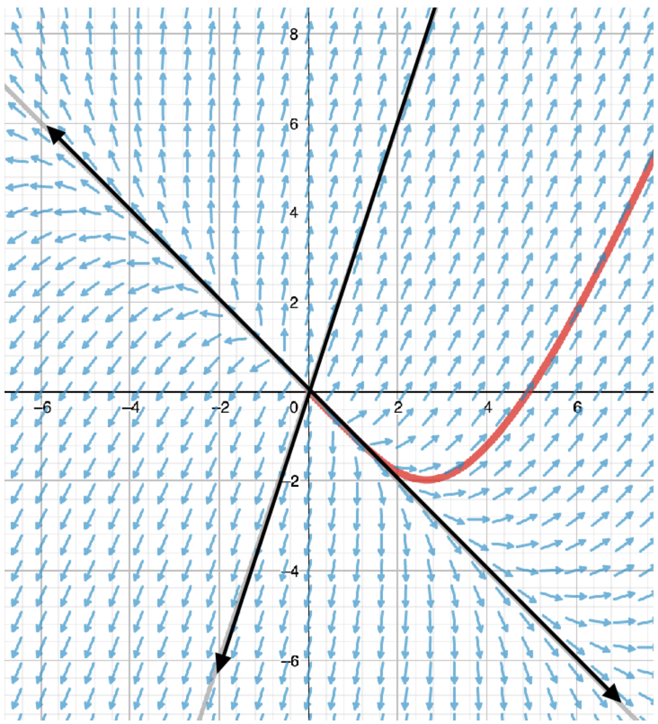
$$\Rightarrow v_1 = -v_2 \quad \text{so} \quad v_{\lambda_1} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\lambda_2 = 5$:

$$A - \lambda_2 I = \begin{pmatrix} -3 & 1 \\ 3 & -1 \end{pmatrix}$$

$$\begin{pmatrix} -3 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow v_1 = 3v_2 \quad \text{so} \quad v_{\lambda_2} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$



$$\vec{x} = c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{t} + c_2\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^{5t}$$

Systems of Ordinary Linear Differential Equations

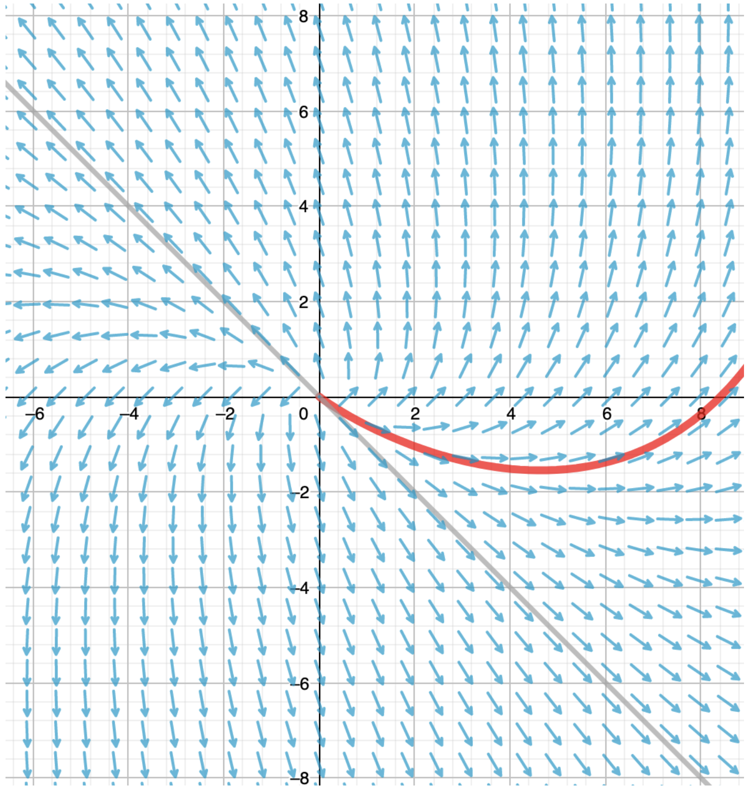
Matrix Solutions

Solve

$$x' = x - y$$
$$y' = x + 3y$$

$$\lambda^2 - 4\lambda + 4 = 0$$

$$\lambda_{1,2} = 2 \qquad \vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$



General Solution:

$$\vec{x}(t) = c_1 \vec{v} e^{\lambda t} + c_2 \left(\vec{w} + \vec{v}\, t\right) e^{\lambda t}$$

where $(A - \lambda I)\vec{w} = \vec{v}$

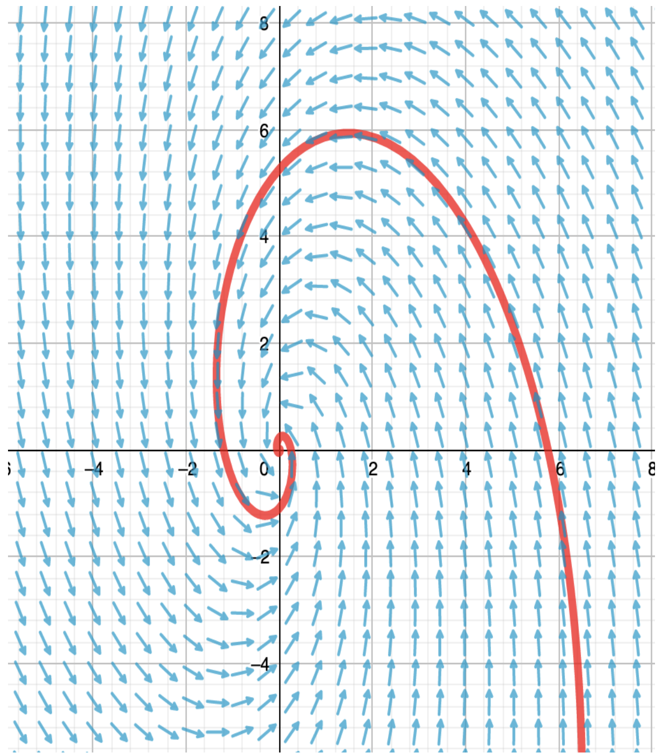
$$\vec{x} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \left(\begin{pmatrix} -1 \\ 0 \end{pmatrix} e^{2t} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} t\, e^{2t} \right)$$

Systems of Ordinary Linear Differential Equations

Matrix Solutions

Solve

$$x' = -x - y$$
$$y' = 4x - y$$



$$\lambda^2 + 2\lambda + 5 = 0$$

$$\lambda_1 = -1 + 2i,\ \lambda_2 = -1 - 2i \qquad \vec{v_1} = \begin{pmatrix} 1 \\ -2i \end{pmatrix},\ \vec{v_2} = \begin{pmatrix} 1 \\ 2i \end{pmatrix}$$

$$\alpha = -1, \qquad \beta = 2$$

General Solution:

$$\vec{x}(t) = c_1 \mathrm{Re}\left\{\vec{v_1}\,(\mathrm{cis}\,\beta t)\right\} e^{\alpha t} + c_2 \mathrm{Im}\left\{\vec{v_2}\,(\mathrm{cis}\,\beta t)\right\} e^{\alpha t}$$

where $\lambda = \alpha \pm \beta i$

$$\vec{x} = c_1 \begin{pmatrix} \cos 2t \\ 2\sin 2t \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -\sin 2t \\ 2\cos 2t \end{pmatrix} e^{-t}$$

Systems of Ordinary Linear Differential Equations

Cauchy and Celestial Mechanics

The proper mathematical history of the eigenvalue begins with celestial mechanics, in particular with Augustin-Louis Cauchy's 1829 paper "Sur l'équation à l'aide de laquelle on détermine les inégalités séculaires des mouvements des planétes" ("On the equation which helps one determine the secular inequalities in the movements of the planets"). This paper, which later appeared in his *Exercises de mathématiques* [Cau1], concerned the observed motion of the planets. At this time, astronomers and mathematicians were making detailed observations of the planets in order to validate the mathematical models inherited from Kepler and Newton's laws of motion. One famous outcome of this program was Urbain Le Verrier's prediction of the existence of Neptune based on observed perturbations in the orbit of Uranus.

Considérons n équations différentielles du premier ordre linéaires et à coefficients constants, entre n variables principales

$$\xi,\ \eta,\ \zeta,\ldots$$

considérées comme fonctions d'une seule variable indépendante t qui pourra désigner le temps. Supposons ces équations présentées sous une forme telle qu'elles fournissent respectivement les valeurs de

$$\frac{d\xi}{dt},\ \frac{d\eta}{dt},\ \frac{d\zeta}{dt},\ldots;$$

de sorte qu'en faisant passer tous les termes dans les premiers membres, on les réduise à

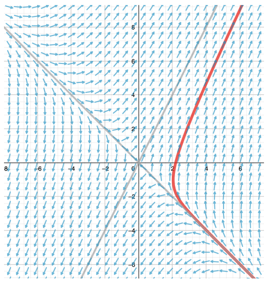
$$(1)\qquad \begin{cases} \dfrac{d\xi}{dt} + \mathcal{L}\xi + \mathfrak{M}\eta + \ldots = 0, \\ \dfrac{d\eta}{dt} + \mathfrak{P}\xi + \mathfrak{Q}\eta + \ldots = 0, \\ \text{etc.,} \end{cases}$$

ou, ce qui revient au même, à

$$(2)\qquad \begin{cases} (D_t + \mathcal{L})\xi + \mathfrak{M}\eta + \ldots = 0, \\ \mathfrak{P}\xi + (D_t + \mathfrak{Q})\eta + \ldots = 0, \\ \text{etc.,} \end{cases}$$

Systems of Ordinary Linear Differential Equations

Solve the following systems, and display the phase protrait along with the associated eigenvectors.



1. $$\frac{dx}{dt} = x + 2y$$ $$\frac{dy}{dt} = 4x + 3y$$

2. $$\frac{dx}{dt} = 2x + 2y$$ $$\frac{dy}{dt} = x + 3y$$

3. $$\frac{dx}{dt} = 3x - y$$ $$\frac{dy}{dt} = 9x - 3y$$

4. $$\frac{dx}{dt} = -6x + 5y$$ $$\frac{dy}{dt} = -5x + 4y$$

5. $$\frac{dx}{dt} = 6x - y$$ $$\frac{dy}{dt} = 5x + 2y$$

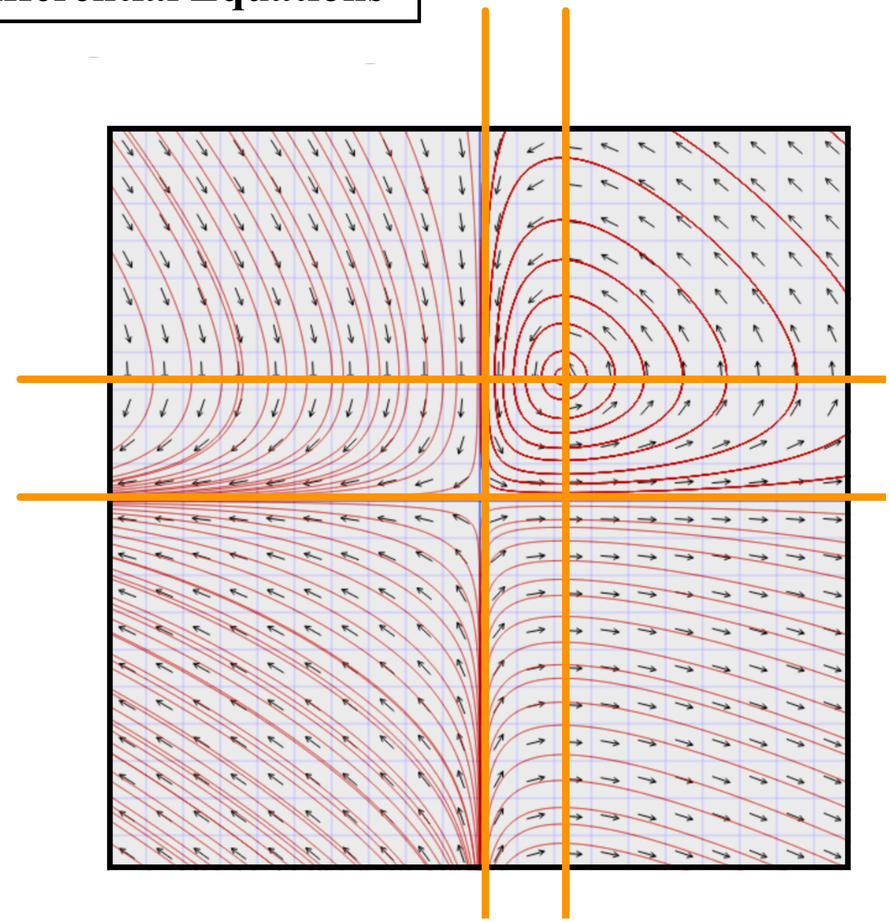
6. $$\frac{dx}{dt} = x + y$$ $$\frac{dy}{dt} = -2x - y$$

Systems of Ordinary Nonlinear Differential Equations

Interacting Species
(Predator-Prey)

$$\frac{dx}{dt} = 2x - 1.2xy$$

$$\frac{dy}{dt} = -y + 0.9xy$$

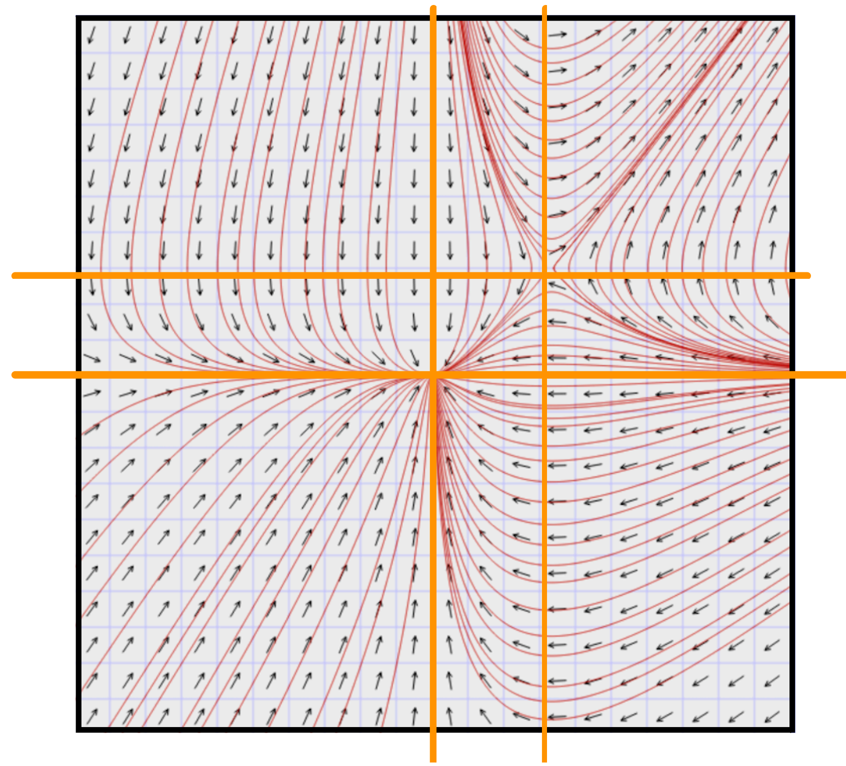


Systems of Ordinary Nonlinear Differential Equations

Interacting Species (Cooperation)

$$\frac{dx}{dt} = -3x + 2xy$$

$$\frac{dy}{dt} = -5y + 3xy$$



Systems of Ordinary Nonlinear Differential Equations

Interacting Species (Competition)

$$\frac{dx}{dt} = 9x - 2x^2 - 4xy$$

$$\frac{dy}{dt} = 8y - 3y^2 - 5xy$$