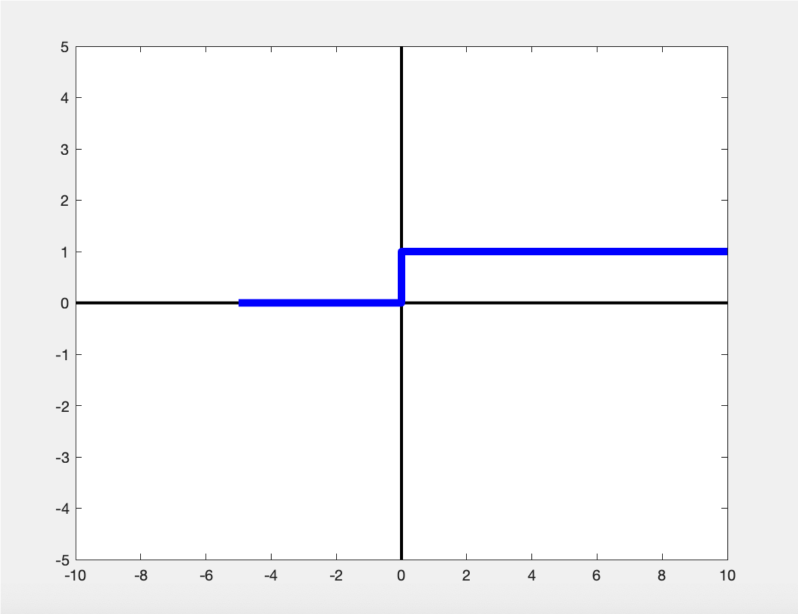
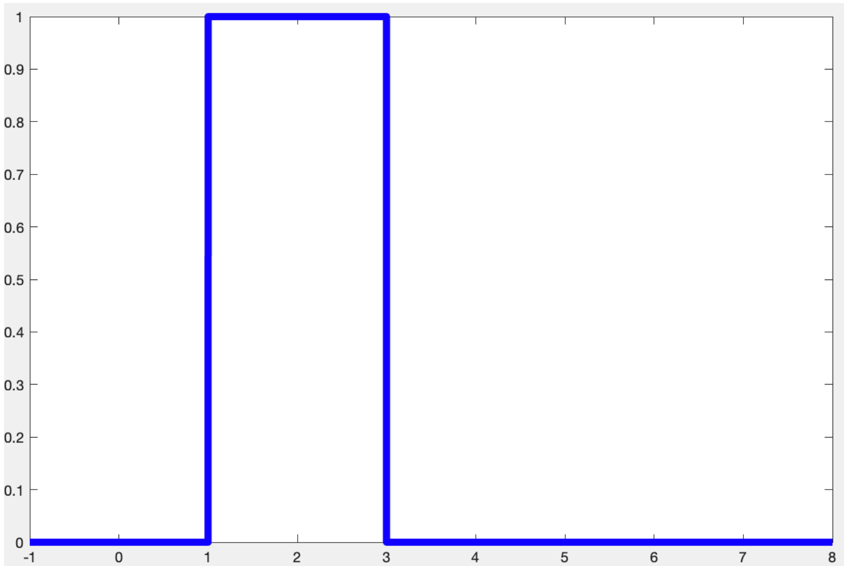
Step Functions



$$u(t-c)=\begin{cases} 0 & t<c \\ 1 & t\geq c \end{cases}$$

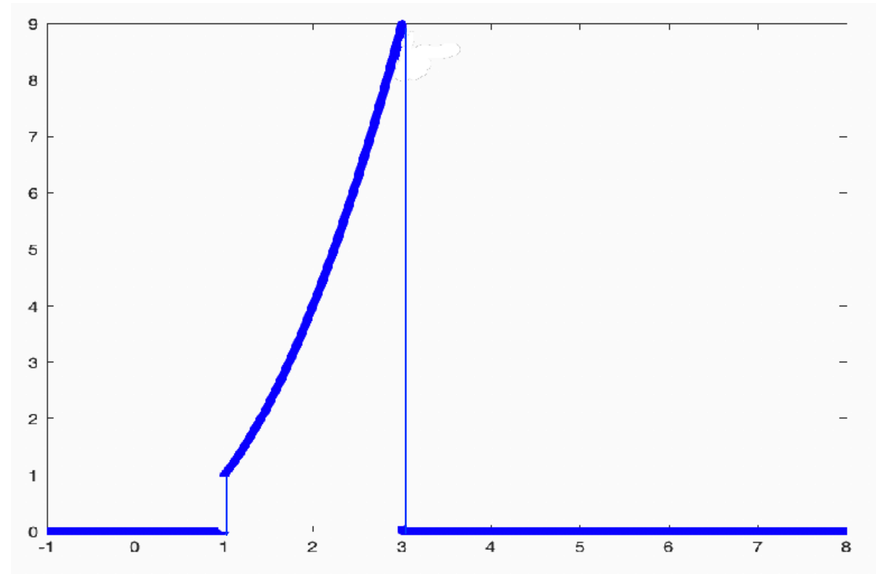
```
syms x
fplot((heaviside(x)) , [-5,10],'b','LineWidth',5)
```

Step Functions



$f(t) = u(t-1) - u(t-3)$

```
syms x
fplot( heaviside(x-1)+heaviside(x-3), [-1, 8],'b','LineWidth',5)
```



$f(t) = t^2(u(t-1) - u(t-3))$

```
syms x
fplot( x^2*(heaviside(x-1)+heaviside(x-3))),
        [-1, 8],'b','LineWidth',5)
```

$$f(t)\left[u(t-a)-u(t-b)\right]=\begin{cases} 0 & t<a \\ f(t) & a<t\le b \\ 1 & t\ge b \end{cases}$$

Step Functions

$$f(t) = \begin{cases} 1 & (0 \le t < 1) \\ 4t - t^2 & (1 \le t < 2) \\ 1 & (2 \le t) \end{cases}$$

$$u(t-0) - u(t-1) + (4t - t^2)[u(t-1) - u(t-2)] + u($$

$$u(t) + u(t-1)(4t - t^2 - 1) - u(t-2)(4t - t^2 + 1)$$

$$\mathcal{L}\{u(t-c)\} = \frac{e^{-cs}}{s}$$

$$\mathcal{L}\{f(t-c)u(t-c)\} = e^{-cs}F(s)$$

$$\mathcal{L}^{-1}\{e^{-cs}F(s)\} = f(t-c)u(t-c)$$

$$\mathcal{L}\{1 - 2u(t-1) + u(t-2)\} = \frac{1}{s} - 2\frac{e^{-s}}{s} + \frac{e^{-2s}}{s}$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^2+4}\right\} = \frac{1}{2}u(t-3)\sin(2(t-3))$$

Step Functions

Solve

$$y'' + y = u(t-3); \quad y(0) = 0,\ y'(0) = 1$$

$$s^2 Y(s) - s\,y(0) - y'(0) + Y(s) = \mathcal{L}\{u(t-3)\}$$

$$s^2 Y(s) - 1 + Y(s) = \frac{e^{-3s}}{s}$$

$$Y(s) = \frac{e^{-3s} + s}{s(s^2+1)} = \frac{e^{-3s}}{s(s^2+1)} + \frac{1}{(s^2+1)}$$

$$= e^{-3s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right) + \frac{1}{(s^2+1)}$$

$$y(t) = u(t-3)\big(1 - \cos(t-3)\big) + \sin t$$

$$= \sin t + u(t-3) - u(t-3)\cos(t-3)$$

Step Functions

Solve

$$y'' + 2y' + 2y = \begin{cases} 1 & (0 \le t < \pi) \\ 0 & (\pi \le t) \end{cases}$$

$$y(0) = 0$$

$$y'(0) = 0$$

$$\left(s^2 Y(s) - sy(0) - y'(0)\right) + 2\left(sY(s) - y(0)\right) + 2Y(s) = \frac{1 - e^{-\pi s}}{s}$$

$$\begin{aligned} Y(s) &= \frac{1 - e^{-\pi s}}{s(s^2 + 2s + 2)} \\ &= \left(1 - e^{-\pi s}\right)\left\{\frac{1}{2s} - \frac{1}{2}\left(\frac{s + 2}{s^2 + 2s + 2}\right)\right\} \\ &= \left(1 - e^{-\pi s}\right)\left\{\frac{1}{2s} - \frac{1}{2}\left(\frac{(s + 1) + 1}{(s + 1)^2 + 1}\right)\right\} \\ &= \left(1 - e^{-\pi s}\right)\left\{\frac{1}{2s} - \frac{s + 1}{2[(s + 1)^2 + 1]} - \frac{1}{2[(s + 1)^2 + 1]}\right\} \\ &= \left(1 - e^{-\pi s}\right)\left\{\frac{1}{2}\mathfrak{L}\{1\} - \frac{1}{2}\mathfrak{L}\{e^{-t}\cos t\} - \frac{1}{2}\mathfrak{L}\{e^{-t}\sin t\}\right\} \end{aligned}$$

Step Functions

Solve

$$y'' + 2y' + 2y = \begin{cases} 1 & (0 \leq t < \pi) \\ 0 & (\pi \leq t) \end{cases}$$

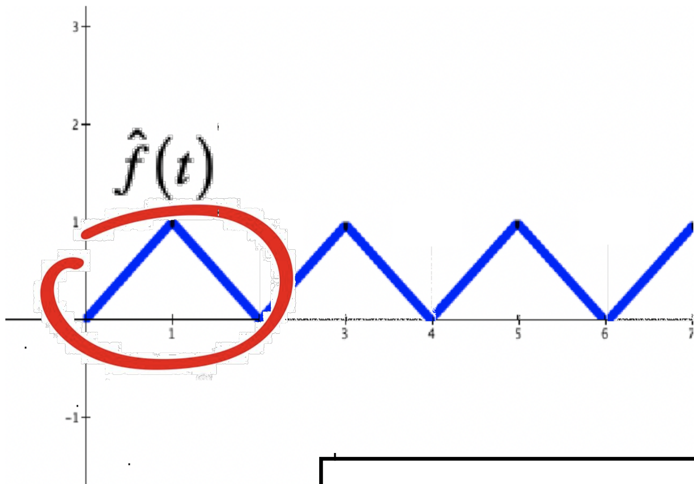
$$y(0) = 0$$

$$y'(0) = 0$$

$$\begin{aligned} y(t) &= \mathfrak{L}^{-1}\{Y(s)\} \\ &= \frac{1}{2} - \frac{1}{2}e^{-t}\cos t - \frac{1}{2}e^{-t}\sin t \\ &\quad - u(t-\pi)\left(\frac{1}{2} - \frac{1}{2}e^{-(t-\pi)}\cos(t-\pi) - \frac{1}{2}e^{-(t-\pi)}\sin(t-\pi)\right) \end{aligned}$$

$$y(t) = \begin{cases} \frac{1}{2}(1 - e^{-t}\cos t - e^{t}\sin t) & (0 \leq t < \pi) \\ \frac{1}{2}e^{-t}(e^{\pi} + 1)(\sin t + \cos t) & (\pi \leq t) \end{cases}$$

Step Functions



$$f(t)=\begin{cases} t & 0 \le t < 1 \\ 2-t & 1 \le t < 2 \end{cases} \quad f(t) \text{ has period } 2$$

$$\mathcal{L}\{f_{periodic}(t)\} = \frac{1}{1-e^{\tau s}}\int_0^{\tau} e^{-st} f(t)\,dt = \frac{1}{1-e^{\tau s}}\mathcal{L}\{\hat{f}(t)\}$$

$$= \frac{1}{1-e^{2s}}\mathcal{L}\{t\,u(t)+2(1-t)u(t-1)+(t-2)u(t-2)\}$$

$$= \frac{1}{1-e^{2s}}\mathcal{L}\{t+2(1-t)u(t-1)+(t-2)u(t-2)\}$$

$$= \frac{1}{1-e^{2s}}\left[\frac{1}{s^2}-2\frac{e^{-s}}{s^2}+\frac{e^{-2s}}{s^2}\right]$$

Impulse Functions

$$\int_0^\infty \delta_a(t) g(t) dt = g(a)$$

$$\mathcal{L}\{\delta_a(t)\} = e^{-as}$$

For problem 1-4 , write in terms of step functions and sketch the graph.

1. $f(t) = \begin{cases} a & (0 \le t < 1) \\ b & (1 \le t < 2) \\ c & (2 \le t) \end{cases}$

2. $f(t) = \begin{cases} 1 & (0 \le t < 1) \\ e^t & (1 \le t < 2) \\ 2 & (2 \le t) \end{cases}$

3. $f(t) = \begin{cases} 1 & (0 \le t < 1) \\ 4t - t^2 & (1 \le t < 2) \\ 1 & (2 \le t) \end{cases}$

4. $f(t) = \begin{cases} 0 & (0 \le t < 1) \\ \sin \pi t & (1 \le t < 2) \\ 0 & (2 \le t) \end{cases}$

For problem 5-14 , determine the Laplace transform of the given function.

5. $1 - u(t-1)$
6. $1 - 2u(t-1) + u(t-2)$
7. $u(t-1)(t-1)$
8. $u(t-2)(t-2)^2$
9. $u(t-\pi)\, sin(t-\pi)$
10. $u(t-3)\, e^3$
11. $u(t)$
12. $u(t-2)t^2$
13. $u(t-\pi) cos\ t$
14. $f(t) = \begin{cases} \sin t & (0 \le t < \pi) \\ 0 & (\pi \le t) \end{cases}$

For problem 15-24 , determine the inverse Laplace transform of the given function.

15. $\dfrac{e^{-s}}{s}$
16. $\dfrac{e^{-s}}{s^2}$
17. $\dfrac{e^{-2s}}{s-3}$
18. $\dfrac{e^{-3s}}{s^2+4}$
19. $\dfrac{e^{-4s}}{s+4}$
20. $\dfrac{e^{-4s}}{(s+4)^2}$
21. $\dfrac{e^{-s}}{s(s+1)}$
22. $\dfrac{s+e^{-\pi s}}{s^2+1}$
23. $\dfrac{e^{-s} - 2e^{-2s} + 2e^{-3s} - e^{-4s}}{s}$
24. $\dfrac{e^{-s} - 2e^{-2s} + 2e^{-3s} - e^{-4s}}{s^2}$

For Problems 1–12, find the solution to the given initial value problem.

1. $y' = 1 - u(t-1)$

$y(0) = 0$

2. $y' = 1 - 2u(t-1) + u(t-2)$

$y(0) = 0$

3. $y' + y = u(t-1)$

4. $y'' + y = t - u(t-1)t$

$y(0) = 0$

$y'(0) = 0$

5. $y'' = 1 - u(t-1)$

$y(0) = 0$

$y'(0) = 0$

6. $y'' + y = u(t-3)$

$y(0) = 0$

$y'(0) = 1$

7. $y'' + y = u(t-\pi) - u(t-2\pi)$

$y(0) = 0$

$y'(0) = 0$

8. $y'' + 4y = t - u\left(t - \frac{\pi}{2}\right)\left(t - \frac{\pi}{2}\right)$

$y(0) = 0$

$y'(0) = 0$

9. $y'' + 4y = u(t-2\pi)\sin t$

$y(0) = 1$

$y'(0) = 0$

10. $y'' + 4y = \sin t - u(t-2\pi)\sin(t-2\pi)$

$y(0) = 0$

$y'(0) = 0$

11. $y'' + y' + 3y = u(t-2)$

$y(0) = 0$

$y'(0) = 1$

12. $y'' + 2y' + 5y = 10 - 10u(t-\pi)$

$y(0) = 0$

$y'(0) = 0$