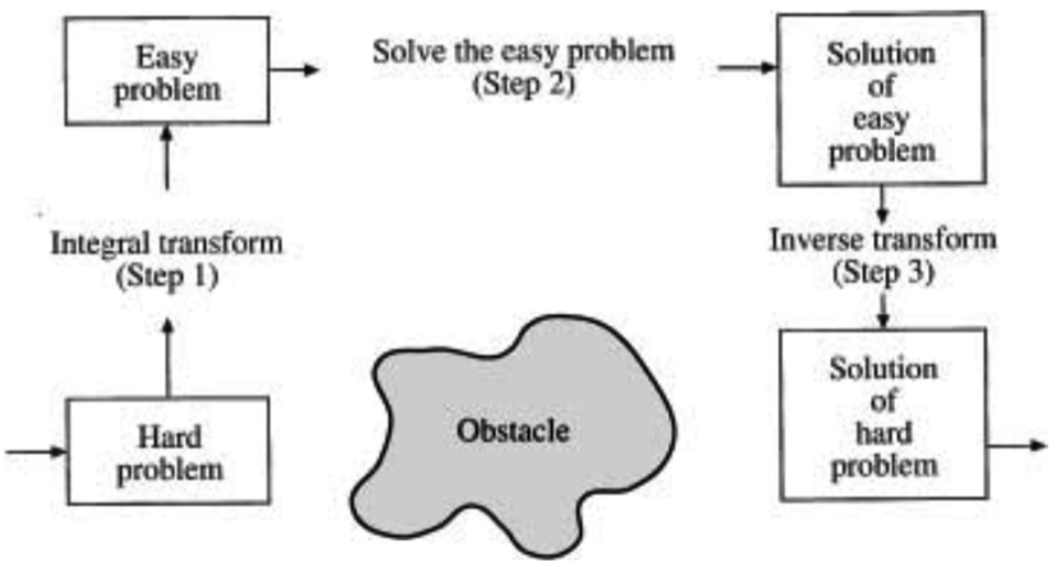
Laplace Transforms



Pierre-Simon Laplace
(1749-1827)

Laplace's Principle of Insufficient Reason

$$y' + 3y = e^{-t}\ ; \quad y(0) = 1$$

$$\mathcal{L}\{y' + 3y\} = \mathcal{L}\{e^{-t}\}$$

$$\mathcal{L}\{y'\} + 3\mathcal{L}\{y\} = \mathcal{L}\{e^{-t}\}$$

$$(sY(s) - y(0)) + 3(Y(s)) = \frac{1}{s+1}$$

$$sY(s) - 1 + 3Y(s) = \frac{1}{s+1}$$

$$Y(s) = \frac{s+2}{(s+1)(s+3)}$$

$$Y(s) = \frac{1}{2(s+1)} + \frac{1}{2(s+3)}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{2(s+1)} + \frac{1}{2(s+3)}\right\}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{(s+3)}\right\}$$

$$y(t) = \frac{1}{2}e^{-t} + \frac{1}{2}e^{-3t}$$

Laplace Transforms

$$\sum_{0}^{\infty} a_k x^k = A(x)$$

$$\sum_{0}^{\infty} a(k) x^k = A(x)$$

$$a(k) \longrightarrow A(x)$$

$$1 \mapsto \frac{1}{1-x} \qquad |x|<1$$

$$\frac{1}{k!} \mapsto e^x \qquad \forall\, x$$

k: $0, 1, 2, \ldots$

t: $0 \le t < \infty$

$$\int_{0}^{\infty} a(t) x^t \, dt = A(x)$$

$x = e^{\ln x}$ so $x^t = \left(e^{\ln x}\right)^t$

$0 < x < 1$ would generally guarantee convergence so then

$\ln x < 0$

Let $s = -\ln x$ or $-s = \ln x$

$$\int_{0}^{\infty} f(t) e^{-st} \, dt = F(s)$$

DEFINITION: Laplace Transform

Let $f(t)$ be a function defined on $(0, \infty)$. **The Laplace transform** of $f(t)$ is defined to be the function $F(s)$ given by the integral

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

The domain of the transform $F(s)$ is taken to be all values of s for which the integral exists. The Laplace transform of $f(t)$ is denoted both by $F(s)$ and the alternate notation $\mathfrak{L}\{f\}$.

$$\begin{aligned}
\mathfrak{L}\{1\} &= \int_0^\infty e^{-st} \cdot 1\, dt \\
&= \lim_{b\to\infty} \int_0^b e^{-st}\, dt \\
&= \lim_{b\to\infty} \left(\frac{-e^{-st}}{s} \bigg|_{t=0}^{t=b} \right) \\
&= \lim_{b\to\infty} \left(\frac{1}{s} - \frac{e^{-sb}}{s} \right) \\
&= \frac{1}{s}
\end{aligned}$$

$$\begin{aligned}
\mathfrak{L}\{e^{kt}\} &= \int_0^\infty e^{-st} e^{kt}\, dt \\
&= \int_0^\infty e^{-(s-k)t}\, dt \\
&= \lim_{b\to\infty} \left(\int_0^b e^{-(s-k)t}\, dt \right) \\
&= \lim_{b\to\infty} \left(-\frac{e^{-(s-k)t}}{s-k} \bigg|_{t=0}^{t=b} \right) \\
&= -\frac{1}{s-k} \lim_{b\to\infty} \left(e^{-(s-k)b} - 1 \right) \\
&= \frac{1}{s-k} \quad \text{for} \quad s > k
\end{aligned}$$

$$\mathcal{L}\{\sin at\} = \int_0^\infty e^{-st}\sin at\,dt \qquad \begin{cases} f = e^{-st} & g = -\frac{1}{a}\cos at \\ f' = -s e^{-st}\,dt & g' = \sin at\,dt \end{cases}$$

$$= -\frac{1}{a}e^{-st}\cos at - \frac{s}{a}\int_0^\infty e^{-st}\cos at\,dt \qquad \begin{cases} f = e^{-st} & g = \frac{1}{a}\sin at \\ f' = -s e^{-st}\,dt & g' = \cos at\,dt \end{cases}$$

$$= -\frac{1}{a}e^{-st}\cos at - \frac{s}{a}\left(\frac{e^{-st}}{a}\sin at + \frac{s}{a}\int_0^\infty e^{-st}\sin at\,dt\right)$$

$$\int_0^\infty e^{-st}\sin at\,dt = -\frac{1}{a}e^{-st}\cos at - \frac{s e^{-st}}{a^2}\sin at - \frac{s^2}{a^2}\int_0^\infty e^{-st}\sin at\,dt$$

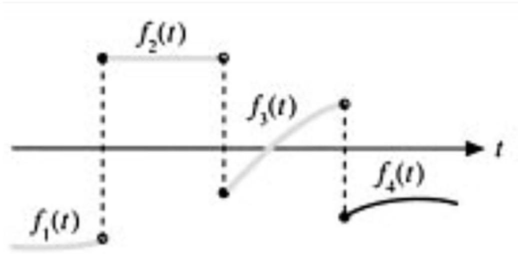
$$\left(1+\frac{s^2}{a^2}\right)\int_0^\infty e^{-st}\sin at\,dt = -\frac{1}{a}e^{-st}\cos at - \frac{s e^{-st}}{a^2}\sin at$$

$$\int_0^\infty e^{-st}\sin at\,dt = \left(\frac{a^2}{a^2+s^2}\right)\left(-\frac{1}{a}e^{-st}\cos at - \frac{s e^{-st}}{a^2}\sin at\right)\Bigg|_0^\infty$$

$$= \frac{-a}{a^2+s^2}e^{-st}\cos at - \frac{s}{a^2+s^2}e^{-st}\sin at\Bigg|_0^\infty$$

$$\lim_{N\to\infty}\left(\frac{-a}{a^2+s^2}e^{-sN}\cos aN - \frac{s}{a^2+s^2}e^{-sN}\sin aN\right) - \left(\frac{-a}{a^2+s^2}e^{-s0}\cos a0 - \frac{s}{a^2+s^2}e^{-s0}\sin a0\right)$$

$$= 0 - \left(\frac{-a}{a^2+s^2} - 0\right) = \frac{a}{a^2+s^2}$$

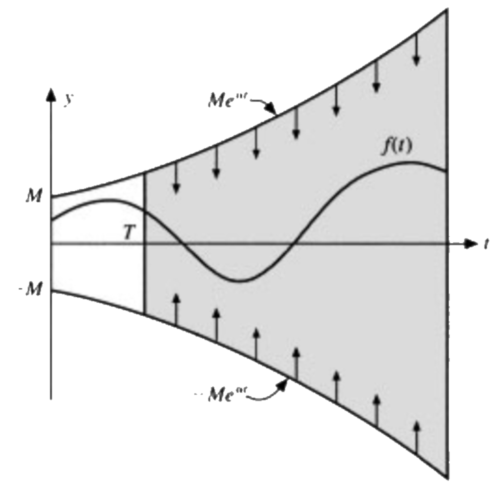


DEFINITION: Exponential Order $e^{\alpha t}$

A function $f(t)$ is said to be of **exponential order** $e^{\alpha t}$ if there exist positive constants M and T such that $f(t)$ satisfies

$$-Me^{\alpha t} \leq f(t) \leq Me^{\alpha t}$$

for all $t \geq T$.



If $f(t)$ is a piecewise continuous function on $[0,\infty)$ and of exponential order $e^{\alpha t}$, then $f(t)$ has a Laplace transform $\mathfrak{L}\{f\} = F(s)$ for $s > \alpha$.

Let f and g be functions whose Laplace transform exists on a common domain. The Laplace transform satisfies the two properties of being a **linear transformation:**

- $\mathfrak{L}\{f+g\} = \mathfrak{L}\{f\} + \mathfrak{L}\{g\}$
- $\mathfrak{L}\{cf\} = c\mathfrak{L}\{f\}$

where c is an arbitrary constant.

$$\begin{aligned}\mathfrak{L}\{5 - 3e^{-2t}\} &= \mathfrak{L}\{5\} + \mathfrak{L}\{-3e^{-2t}\} \\ &= 5\mathfrak{L}\{1\} - 3\mathfrak{L}\{e^{-2t}\} \\ &= 5\left(\frac{1}{s}\right) - 3\left(\frac{1}{s+2}\right) \\ &= \frac{2(s+5)}{s(s+2)} \qquad (s > 0)\end{aligned}$$

$f(t)$	$F(s) = \mathfrak{L}\{f\}$	Domain of $F(s)$
1	$1/s$	$s > 0$
t^n (n positive integer)	$\dfrac{n!}{s^{n+1}}$	$s > 0$
$t^p \quad (p > -1)$	$\dfrac{\Gamma(p+1)}{s^{p+1}}$	$s > 0$
e^{at}	$\dfrac{1}{s-a}$	$s > a$
$e^{at} t^n,\ n = 1, 2, \ldots$	$\dfrac{n!}{(s-a)^{n+1}}$	$s > a$
$\sin bt$	$\dfrac{b}{s^2 + b^2}$	$s > 0$
$\cos bt$	$\dfrac{s}{s^2 + b^2}$	$s > 0$
$\sinh bt$	$\dfrac{b}{s^2 - b^2}$	$s > \lvert b \rvert$
$\cosh bt$	$\dfrac{s}{s^2 - b^2}$	$s > \lvert b \rvert$
$e^{at} \sin bt$	$\dfrac{b}{(s-a)^2 + b^2}$	$s > a$
$e^{at} \cos bt$	$\dfrac{s-a}{(s-a)^2 + b^2}$	$s > a$
$u(t-c)$	$\dfrac{e^{-cs}}{s}$	$s > 0$
$u(t-c)f(t-c)$	$e^{-cs}F(s)$	
$\int_0^t f(t-\tau)g(\tau)\, d\tau$	$F(s)G(s)$	
$\delta(t-c)$	e^{-cs}	
$\dfrac{d^n}{dt^n} f(t)$	$s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$	
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	

- **Find the Laplace transforms of the following -**

1. $at^2 + bt + c$
2. $t^2 + e^{2t} - 2$
3. $(t - 9)^2$
4. $e^{2t - 1}$
5. $(1 + e^t)^2$
6. $3\, t \sin t$
7. $t^2 \sin 2t$
8. $e^{-2t} \sin 3t$
9. $5\, e^{5t} \cos 2t$
10. $t^2 e^{-3t}$
11. $tet \cos t$
12. $t^2 e^t \sin t$
13. $\sin 2t \sinh 2t$
14. $\sin^2 t$
15. $\sinh 3t$
16. $\cos^3 t$
17. $t \sin^2 t$
18. $\cos mt \sin nt \ (m \neq n)$

- **Find the Inverse Laplace transforms for the following -**

1. $\frac{1}{s^3}$
2. $\frac{2}{s} + \frac{3}{s - 1} + \frac{7}{s^3}$
3. $\frac{5}{s^2 + 3}$
4. $\frac{3}{s - 3} + \frac{4}{s + 3}$
5. $\frac{1}{s^2 + 3s}$
6. $\frac{s + 1}{s^2 + 2s + 10}$
7. $\frac{1}{s^2 + 4s + 4}$
8. $\frac{1}{s^2 + 4s + 4}$
9. $\frac{s + 1}{s^2 + s - 2}$
10. $\frac{5}{s^2 + s - 6}$
11. $\frac{2s + 4}{s^2 - 1}$
12. $\frac{7}{(s + 2)^2 + 3}$
13. $\frac{2s + 16}{s^2 + 4s + 13}$
14. $\frac{6}{(s + 2)^4}$
15. $\frac{7s^2 + 23s + 30}{(s - 2)(s^2 + 2s + 5)}$
16. $\frac{4}{s^2(s^2 + 4)}$
17. $\frac{3}{(s^2 + 1)(s^2 + 4)}$
18. $\frac{7s - 1}{(s + 1)(s + 2)(s - 3)}$
19. $\frac{s^2 - 2}{s^3 + 8s^2 + 7s}$
20. $\frac{s^2 + 9s + 2}{(s - 1)^2 (s + 3)}$

$$y' + 3y = e^{-t} \; ; \quad y(0) = 1$$

$$\mathcal{L}\{ y' + 3y \} = \mathcal{L}\{ e^{-t} \}$$

$$\mathcal{L}\{ y' \} + 3\mathcal{L}\{ y \} = \mathcal{L}\{ e^{-t} \}$$

$$\left(sY(s) - y(0) \right) + 3\left(Y(s)\right) = \frac{1}{s+1}$$

$$sY(s) - 1 + 3Y(s) = \frac{1}{s+1}$$

$$Y(s) = \frac{s+2}{(s+1)(s+3)}$$

$$Y(s) = \frac{1}{2(s+1)} + \frac{1}{2(s+3)}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{ \frac{1}{2(s+1)} + \frac{1}{2(s+3)} \right\}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \frac{1}{2}\mathcal{L}^{-1}\left\{ \frac{1}{(s+1)} \right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{ \frac{1}{(s+3)} \right\}$$

$$y(t) = \frac{1}{2}e^{-t} + \frac{1}{2}e^{-3t}$$

- **Use Laplace transforms to solve the problems below.**

1. $y' = 1$	$y(0) = 1$	
2. $y' - y = 0$	$y(0) = 1$	
3. $y' - y = e^t$	$y(0) = 1$	
4. $y' + y = e^{-t}$	$y(0) = 1$	
5. $y'' = e^t$	$y(0) = 1$	$y'(0) = 1$
6. $y'' - 3y' + 2y = 0$	$y(0) = 1$	$y'(0) = 1$
7. $y'' + 2y' = 4$	$y(0) = 1$	$y'(0) = -4$
8. $y'' + 9y = 20e^{-t}$	$y(0) = 1$	$y'(0) = 1$
9. $y'' + 9y = \cos 3t$	$y(0) = 1$	$y'(0) = -1$
10. $y'' + 4y = 4$	$y(0) = 1$	$y'(0) = 1$
11. $y'' - 2y' + 5y = 0$	$y(0) = 2$	$y'(0) = 4$
12. $y'' + 10y' + 25y = 0$	$y(0) = 0$	$y''(0) = 10$
13. $y'' + 3y' + 2y = 6$	$y(0) = 1$	$y'(0) = 2$
14. $y'' + y = \sin t$	$y(0) = 2$	$y'(0) = -1$

$$y'' + 4y = \sin t\,; \quad y(0) = 0,\; y'(0) = 1$$

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{\sin t\}$$

$$s^2Y(s) - s\,y(0) - y'(0) + 4Y(s) = \frac{1}{s^2+1}$$

$$Y(s) = \frac{s^2+2}{(s^2+1)(s^2+4)} = \frac{\frac{1}{3}}{(s^2+1)} + \frac{\frac{2}{3}}{(s^2+4)}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)}\right\} + \frac{1}{3}\mathcal{L}^{-1}\left(\frac{2}{(s^2+4)}\right)$$

$$y(t) = \frac{1}{3}\sin t + \frac{1}{3}\sin 2t$$

$$t\,y'' - 2t\,y' + 4\,y = 0\,; \quad y(0) = 0,\ y'(0) = -1$$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\left(\mathcal{L}\{f(t)\}\right)$$

$$\mathcal{L}\{t^1 y''\} = (-1)\frac{d}{ds}\left(\mathcal{L}\{y''\}\right) = -\frac{d}{ds}\left(s^2 Y(s) - sy(0) - y'(0)\right)$$

$$= -\frac{d}{ds}\left(s^2 Y(s) + 1\right)$$

$$= -2sY(s) - s^2 Y'(s)$$

$$\mathcal{L}\{t^1 y'\} = (-1)\frac{d}{ds}\left(\mathcal{L}\{y'\}\right) = -\frac{d}{ds}\left(sY(s) - y(0)\right)$$

$$= -Y(s) - sY'(s)$$

$$-2sY(s) - s^2 Y'(s) + 2Y(s) + 2\,sY'(s) + 4Y(s) = 0$$

$$\left(2s - s^2\right)Y'(s) + (6 - 2s)Y(s) = 0$$

$$\left(2s - s^2\right)\frac{dY}{ds} + (6 - 2s)Y = 0$$

$$\frac{1}{Y}dY = \frac{2s - 6}{s(2 - s)}ds$$

$$\int \frac{1}{Y}dY = \int \frac{2s - 6}{s(2 - s)}ds$$

$$\ln|\,Y(s)\,| = \ln\left|\,\frac{s - 2}{s^3}\,\right| + C$$

$$Y(s) = C\left(\frac{1}{s^2} - \frac{2}{s^3}\right)$$

$$\mathcal{L}^{-1}\{Y(s)\} = C\mathcal{L}^{-1}\left\{\frac{1}{s^2} - \frac{2}{s^3}\right\}$$

$$y(t) = C\left(t - t^2\right)$$

$$y(t) = t^2 - t$$