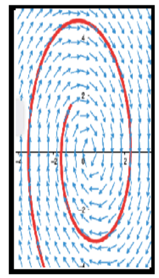
$$F = ma$$

$$f(t) \rightarrow$$

$$kx \rightarrow$$

$$c_v \dot{x} \rightarrow$$



$$\rightarrow m\ddot{x}$$

$$-kx - c_v \dot{x} + f(t) = m\ddot{x}$$

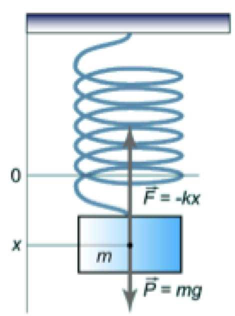
$$A\frac{d^2y}{dt^2} + B\frac{dy}{dt} + Cy = f(t)$$

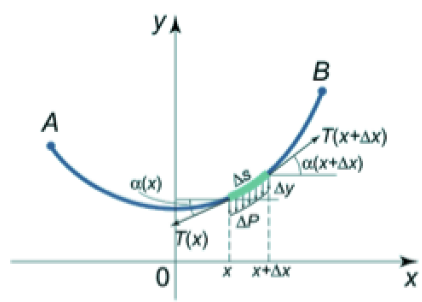
acceleration
curve bending

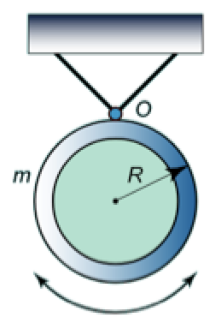
damping
friction
resistance

restoring force

"forcing" function







mth_211

$$Ay'' + By' + Cy = 0$$

constant coefficients homogeneous

If e^{st} is a solution to $Ay'' + By' + Cy = 0$ then $A\left(e^{st}\right)'' + B\left(e^{st}\right)' + Ce^{st} = 0$

$As^2e^{st} + Bse^{st} + Ce^{st} = 0$

$e^{st}\left(As^2 + Bs + C\right) = 0$

$As^2 + Bs + C = 0$ is called the **characteristic equation** and $c_1 e^{s_1 t}$ and $c_2 e^{s_2 t}$ are solutions. The general solution can the be represented as $y = c_1 e^{s_1 t} + c_2 e^{s_2 t}$.

- $y'' - 3y' + 2y = 0 \quad ; \quad y(0) = 1,\ y'(0) = 0$
- $y'' - 25y = 0$
- $3y'' + 2y' + \frac{1}{4}y = 0$
- $y'' - y' - y = 0$
- $y'' + 6y' + 9y = 0$

$$Ay'' + By' + Cy = 0$$

constant coefficients homogeneous

If $s_1 = s_2$ then the general solution is $y = c_1 e^{s_1 t} + c_2\, t\, e^{s_2 t}$. $Ay'' + By' + Cy = 0$ then

$y = As^2 + Bs + C = 0$ and $\dfrac{dy}{ds} = 2As + B = 0$

Let $t\,e^{st}$ be a solution.

$y = t\,e^{st}$

$y' = e^{st} + s\,te^{st} = e^{st} + s\,y$

$y'' = s\,e^{st} + sy' = s\,e^{st} + s\left(e^{st} + s\,y\right) = s^2 y + 2s\,e^{st}$

$A\left(s^2 y + 2s\,e^{st}\right) + B\left(s\,y + e^{st}\right) + Cy = 0$

and $\left(As^2 + Bs + C\right)y + \left(2As + B\right)e^{st}$ is zero !

- $y'' + 6y' + 9y = 0$

$$Ay'' + By' + Cy = 0$$

constant coefficients

homogeneous

- $y'' - 4y' + 9y = 0$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} \cdots$$

So

$$\begin{aligned}
e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \cdots \\
&= 1 + ix - \frac{(x)^2}{2!} - \frac{i(x)^3}{3!} + \frac{(x)^4}{4!} + \frac{i(x)^5}{5!} - \frac{(x)^6}{6!} - \frac{i(x)^7}{7!} + \cdots \\
&= \left(1 - \frac{(x)^2}{2!} + \frac{(x)^4}{4!} - \frac{(x)^6}{6!} + \cdots\right) + i\left(x - \frac{(x)^3}{3!} + \frac{(x)^5}{5!} - \frac{(x)^7}{7!} + \cdots\right) \\
&= \cos x + i \sin x
\end{aligned}$$

$$\underbrace{A}y'' + \underbrace{B}y' + \underbrace{C}y = \underbrace{0}_{\text{homogeneous}}$$

constant coefficients

- $y'' - 4y' + 9y = 0$

$s_1 = 2 + i\sqrt{5}\,,\ s_2 = 2 - i\sqrt{5}$

$s_1 = \alpha + \beta i\,,\ s_2 = \alpha - \beta i$

$$y(t) = c_1 e^{(\alpha+\beta i)t} + c_2 e^{(\alpha-\beta i)t} = y(t) = c_1 e^{\alpha t} e^{i(\beta t)} + c_2 e^{\alpha t} e^{-i(\beta t)}$$

$$y(t) = c_1 e^{\alpha t}(\cos\beta t + i\sin\beta t) + c_2 e^{\alpha t}(\cos\beta t - i\sin\beta t)$$

$$y(t) = (c_1 + c_2)e^{\alpha t}(\cos\beta t) + (ic_1 - ic_2)e^{\alpha t}(\sin\beta t)$$

$$y(t) = c_1 e^{\alpha t}(\cos\beta t) + c_2 e^{\alpha t}(\sin\beta t)$$

$$y(t) = e^{\alpha t}(c_1\cos\beta t + c_2\sin\beta t)$$

General Statement

If $s_1 = \alpha + \beta i$ and $s_2 = \alpha - \beta i$

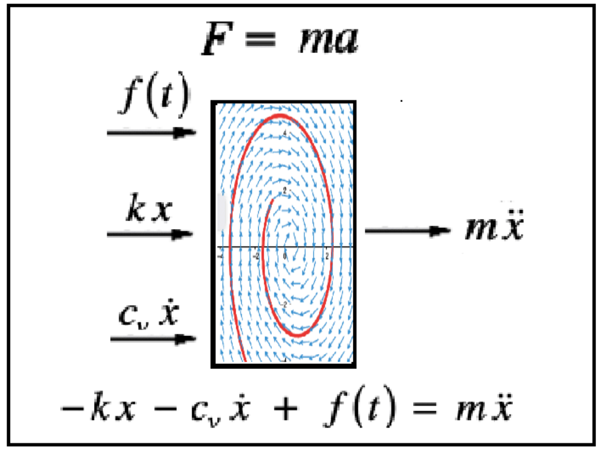
$$Ay'' + By' + Cy = 0$$

then

$$y(t) = e^{\alpha t}(c_1\cos\beta t + c_2\sin\beta t)$$

Solve the following for *y*

1. $y'' - 4y = 0$
2. $y'' = 0$
3. $y'' - 4y' + 4y = 0$
4. $y'' + y' - 2y = 0$
5. $5y'' - 10y' = 0$
6. $\frac{2}{3}y'' + 4y' + 6y = 0$
7. $y'' + y = 0$
8. $y'' + 4y' - y = 0$
9. $y'' + 2y' + y = 0$
10. $y'' + 2y' + 101y = 0$
11. $y'' - 2y' + 101y = 0$
12. $y'' + 4\pi^2 y' + 6y = 0$
13. $y'' + y = 0;\ y(0) = 0,\ y'(0) = 2$
14. $y'' + 3y' + 2y = 0;\ y(0) = 0,\ y'(0) = 1$
15. $y'' - 9y = 0;\ y(0) = 2,\ y'(0) = -1$
16. $y'' - 6y = 0;\ y(0) = 1,\ y'(0) = -1$
17. $y'' - 25y = 0;\ y(1) = 0,\ y'(1) = 1$



$$A\frac{d^2y}{dt^2} + B\frac{dy}{dt} + Cy = f(t)$$

acceleration
curve bending

restoring force

$$A\frac{d^2y}{dt^2} + Cy = 0$$

<----- NO "forcing" function

acceleration
curve bending

restoring force

Free Harmonic Motion

$$A\frac{d^2y}{dt^2} + Cy = 0$$ <----- NO "forcing" function

acceleration
curve bending

restoring force

$$m\,y'' + ky = 0$$

$$y_1(t) = c_1 \cos\sqrt{\frac{k}{m}}\,t \qquad y_2(t) = c_2 \sin\sqrt{\frac{k}{m}}\,t$$

Let $\omega = \sqrt{\frac{k}{m}}$, then $y(t) = c_1 \cos\omega t + c_2 \sin\omega t$

If $y(0) = 0$, then $c_1 = y(0)$ and $c_2 = \frac{y'(0)}{\omega}$

and

Free Harmonic Motion

$$A\frac{d^2y}{dt^2} \quad + \quad Cy = \quad 0$$

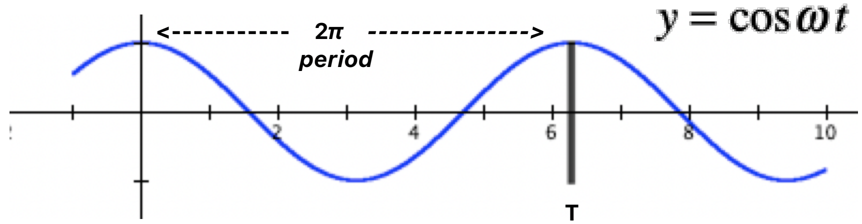
<----- NO "forcing" function

acceleration
curve bending

restoring force

$$y(t) = y(0)\cos\omega t + \frac{y'(0)}{\omega}\sin\omega t$$

$$y = \cos\omega t$$



To convert a cycle per second (hertz) measurement to a radian per second measurement, **multiply the frequency by th conversion ratio**. The frequency in radians p second is equal to the cycles per second multiplied by 6.283185.

$$\omega T = 2\pi$$

$$T = \frac{2\pi}{\omega}$$

radians per second

$$f = \frac{1}{T} \quad \text{or} \quad fT = 1$$

$$y(t) = c_1 e^{i\omega t} + c_2 e^{-i\omega t} \qquad y(t) = y(0)e^{i\omega t} + \frac{y'(0)}{\omega}e^{-i\omega t}$$

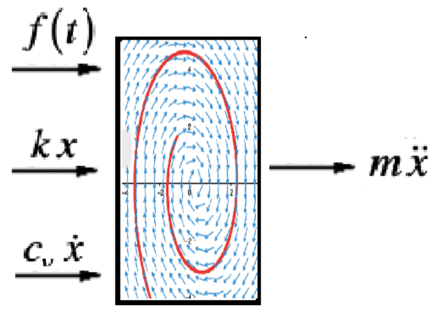
π = 6.283185 ...

XI. Sit CAB bullula aerea, quoad fieri poteſt compreſſa, quae proin eſt materia ſubtili vorticoſa penitus repleta. Circumdata vero ſit cruſta aquea ADEB, vt ergo reliquum ſpatium CDE materia ſubtili impleatur. Sit AC=g, CD=b. Sumatur pro ratione radii ad peripheriem, 1 : π, pro grauitate ſpecifica materiae ſubtilis, n et pro grauitate ſpecifica aquae ſeu cruſtae *Fig. I.*

$$e^{\pi i} = -1 \qquad e^{\frac{\pi}{2} i} = 1 \qquad e^{\tau i} = 1$$

Par soit π la circonference d'un cercle dont le rayon est = 1,

$$F = ma$$



$$-kx - c_v \dot{x} + f(t) = m\ddot{x}$$

$$A \frac{d^2x}{dt^2} + B\frac{dx}{dt} + Cx = 0$$

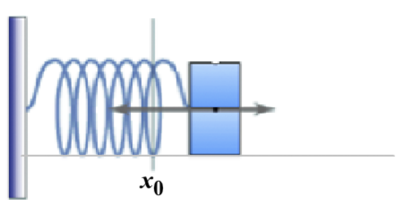
<----- NO "forcing" function

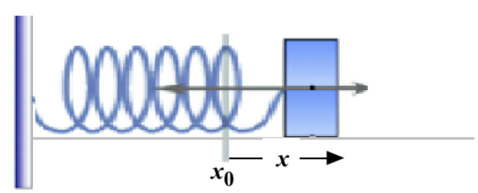
restoring force (Hooke's law)

friction

acceleration
curve bending

Unforced Damped Motion





Unforced Undamped Motion

$$A\frac{d^2x}{dt^2} + \qquad + \; Cx = 0$$

<----- NO "forcing" function

restoring force (Hooke's law)

NO friction

acceleration
curve bending

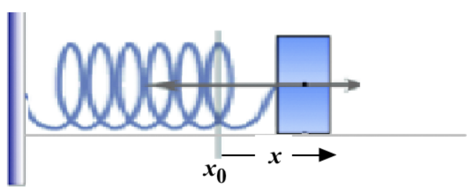
$$As^2 + Bs + C = 0$$

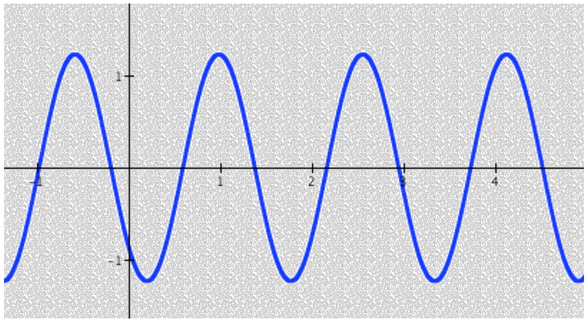
$$s_1, s_2 = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

If $B = 0, \quad s = \pm i\omega$

$$x(t) = c_1 \cos\omega t + c_2 \sin\omega t$$

- **Undamped Motion**





Unforced Damped Motion

$$A\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Cx = 0$$

<----- NO "forcing" function

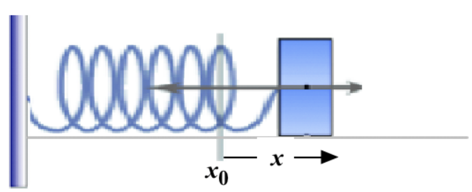
restoring force (Hooke's law)

friction

acceleration
curve bending

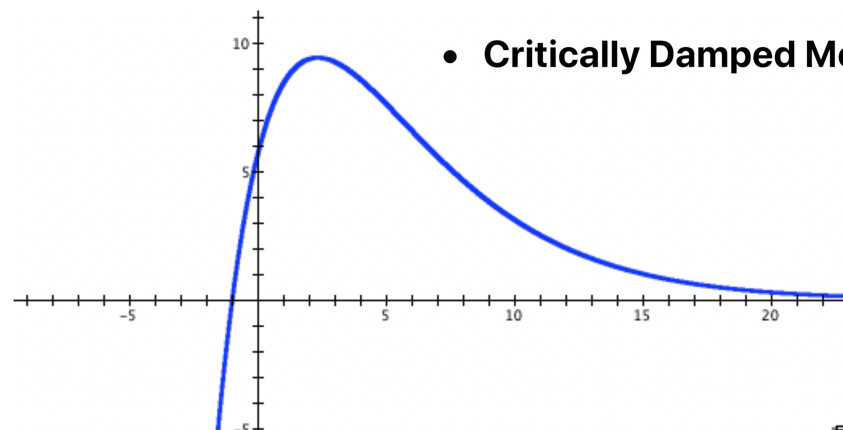
$$As^2 + Bs + C = 0$$

$$s_1,\, s_2 = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$



If $B^2 - 4AC = 0, \quad s = s_1 = s_2 = \dfrac{-B}{2A}$

$$x(t) = c_1 e^{st} + c_2\, t\, e^{st}$$



- **Critically Damped Motion**

Unforced Damped Motion

$$A\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Cx = 0$$

<----- NO "forcing" function

restoring force (Hooke's law)

friction

acceleration
curve bending

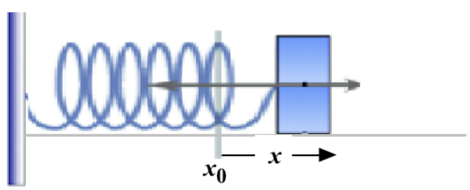
$$As^2 + Bs + C = 0$$

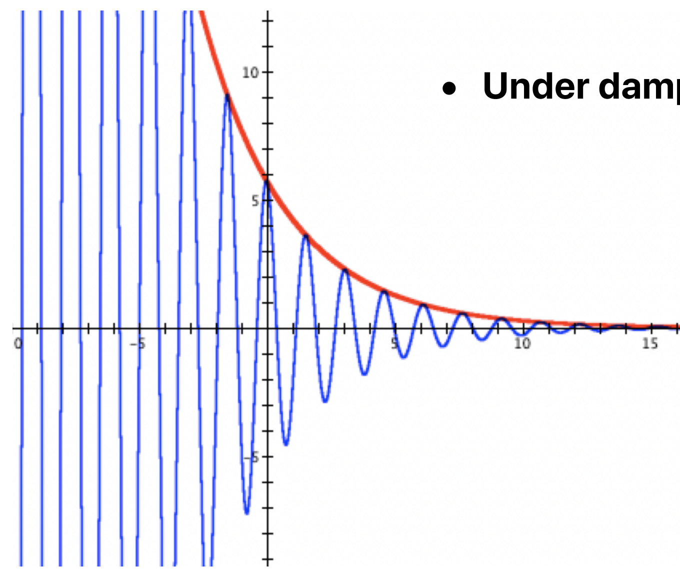
$$s_1, s_2 = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

If $B^2 < 4AC$, $\quad s_1, s_2 = \frac{-B}{2A} \pm \frac{i}{2A}\sqrt{|B^2 - 4AC|}$

$$s_1, s_2 = \alpha \pm \beta i$$

$$x(t) = e^{\alpha t}(c_1 \cos\beta t + c_2 \sin\beta t)$$





- **Under damped Motion**

Unforced Damped Motion

$$A\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Cx = 0$$

<----- NO "forcing" function

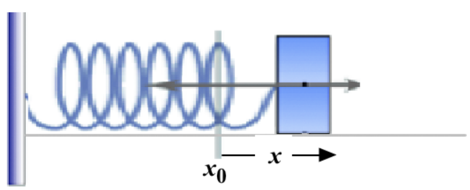
restoring force (Hooke's law)

friction

acceleration
curve bending

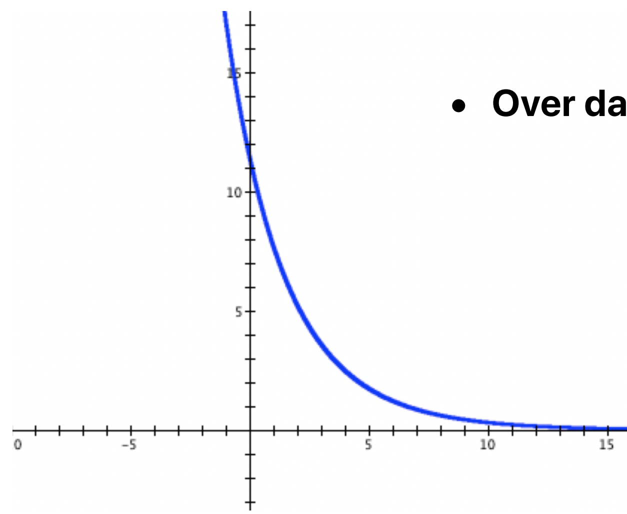
$$As^2 + Bs + C = 0$$

$$s_1, s_2 = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$



If $B^2 > 4AC$, $\quad s_1, s_2 = \dfrac{-B \pm \sqrt{B^2 - 4AC}}{2A}$

$$x(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t}$$



- **Over damped Motion**