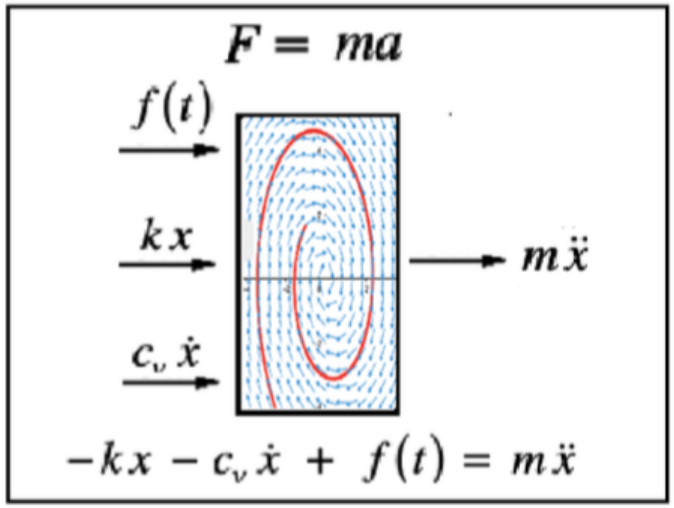



$$A\frac{d^2y}{dt^2} \quad + \quad B\frac{dy}{dt} \quad + \quad Cy = \quad f(t)$$

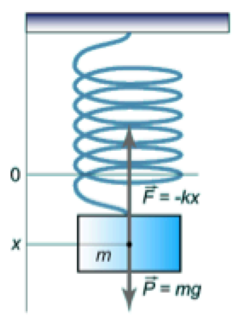
acceleration
curve bending

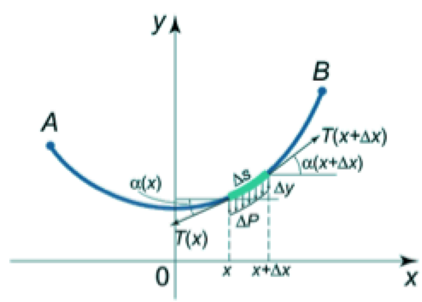
damping
friction
resistance

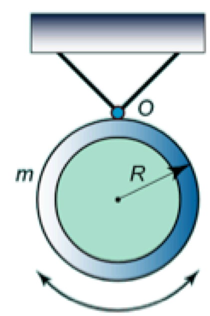
restoring force

"forcing" function

mth_211



**Reduction of order method**

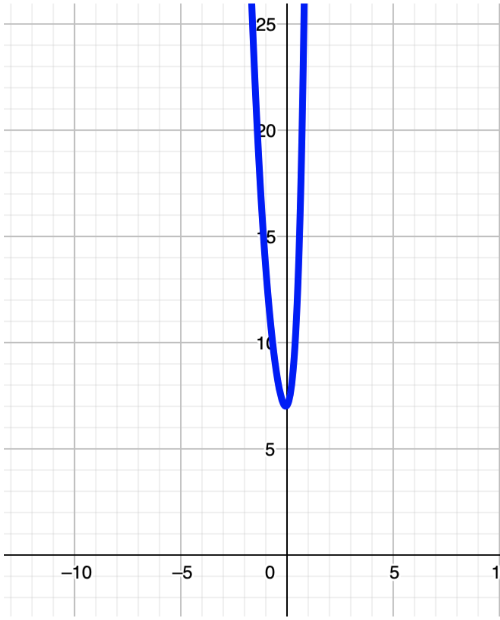
a) Solve $y'' - 2y' - 3y = 0$ given that $y_1(t) = e^{3t}$.

Assume $y_2(t) = e^{3t}v \Rightarrow y_2'(t) = 3e^{3t}v + e^{3t}v'$

$$\Rightarrow y_2''(t) = 9e^{3t}v + 3e^{3t}v' + 3e^{3t}v' + e^{3t}v'' = 9e^{3t}v + 6e^{3t}v' + e^{3t}v''$$

Substituting we have $\left(9e^{3t}v + 6e^{3t}v' + e^{3t}v''\right) - 2\left(3e^{3t}v + e^{3t}v'\right) - 3\left(e^{3t}v\right) = 0$ or

$$e^{3t}v'' + 4e^{3t}v' = 0.$$




Now let $w = v' \Rightarrow w' = v''$. with this substitution we have

$$e^{3t}w' + 4e^{3t}w = 0 \text{ or } w' + 4w = 0.$$

The solution to this first order linear differential equation is $w(t) = ce^{-4t}$.

$$\therefore\ v = \int w\,dt = \int ce^{-4t}\,dt = -\frac{1}{4}ce^{-4t} + k.$$

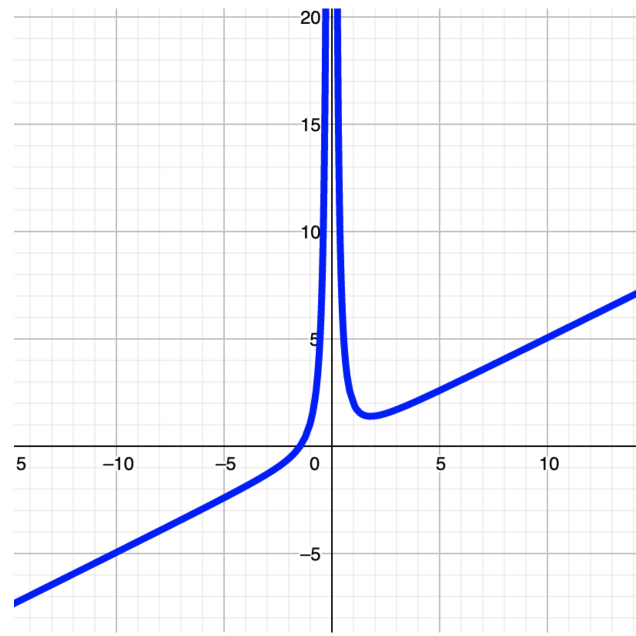
Dropping constants we get $y_2(t) = e^{3t}\left(e^{-4t}\right) = e^{-t}$ and the general solution to our differential equation is

$$y(t) = c_1e^{3t} + c_2e^{-t}$$

**b)** Solve $t^2 y'' + 2t y' - 2y = 0$ given that $y_1(t) = t$ .

Assume $y_2(t) = t v \quad \Rightarrow \quad y_2'(t) = v + t v' \quad \Rightarrow \quad y_2''(t) = v' + v' + t v' = 2v' + t v'$

Substituting we have $t^2(2v' + tv') + 2t(v + tv') - 2(tv) = 0$ or $t^3 v'' + 4t^2 v' = 0$ .




Now let $w = v' \quad \Rightarrow \quad w' = v''$. with this substitution we have

$$t^3 w' + 4t^2 w = 0 .$$

The solution to this first order linear differential equation is $w(t) = c t^{-4}$ .

$$\therefore \; v = \int w \, dt = \int c t^{-4} \, dt = -\frac{1}{3} c t^{-3} + k .$$

Dropping constants we get $y_2(t) = t(t^{-3}) = t^{-2}$ and the general solution to our differential equation is

$$y(t) = c_1 t + \frac{c_2}{t^2}$$

## Reduction of Order

**Use reduction of order to find a particular solution to the following -**

1. $y'' - 4y' + 4y = 0; \quad y_1 = e^{2x}$
2. $y'' + 2y' + y = 0; \quad y_1 = xe^{-x}$
3. $y'' + 16y = 0; \quad y_1 = \cos 4x$
4. $y'' + 9y = 0; \quad y_1 = \sin 3x$
5. $y'' - y = 0; \quad y_1 = \cosh x$
6. $y'' - 25y = 0; \quad y_1 = e^{5x}$
7. $9y'' - 12y' + 4y = 0; \quad y_1 = e^{2x/3}$
8. $6y'' + y' - y = 0; \quad y_1 = e^{x/3}$
9. $x^2y'' - 7xy' + 16y = 0; \quad y_1 = x^4$
10. $x^2y'' + 2xy' - 6y = 0; \quad y_1 = x^2$
11. $xy'' + y' = 0; \quad y_1 = \ln x$
12. $4x^2y'' + y = 0; \quad y_1 = x^{1/2} \ln x$
13. $x^2y'' - xy' + 2y = 0; \quad y_1 = x \sin(\ln x)$

**Taylor Series Expansion**

$$f(x) = f(a) + f'(x-a) + \frac{f''(x-a)^2}{2!} + \frac{f'''(x-a)^3}{3!} + \cdots$$

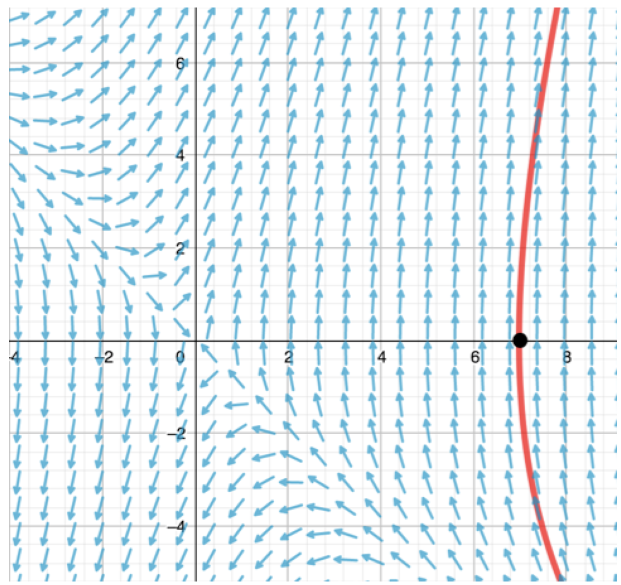
$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x-a)^k}{k!}$$

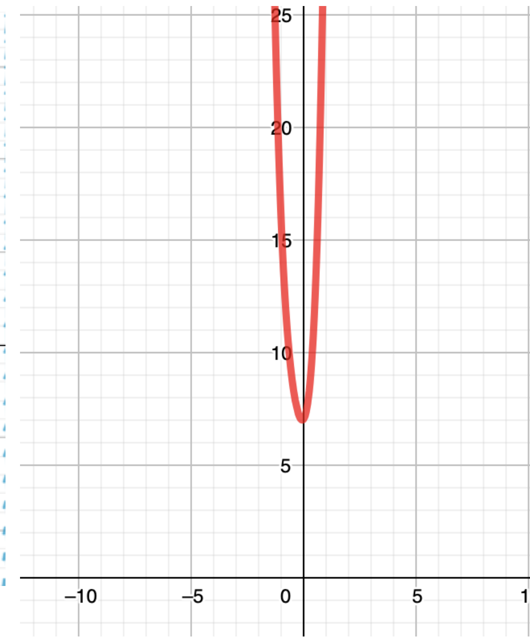
**Solution in power series**

**a)** Find a series solution for $y'' - 2y' - 3y = 0$ such that $y(0) = 7$ and $y'(0) = 1$.

Solve for $y''$ and differentiate

$$y'' = 2y' + 3y \quad \text{and then}$$
$$y''' = 2y'' + 3y'$$
$$y^{iv} = 2y''' + 3y''$$





Evaluate the derivatives at $t = 0$

$$y''(0) = 23$$
$$y'''(0) = 49$$
$$y^{iv}(0) = 167$$

The solution is then

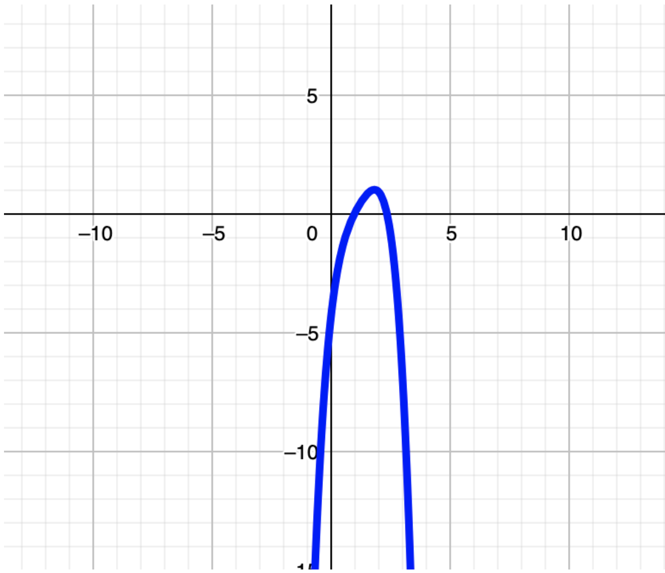
$$y(t) = 7 + 1(t-0)^1 + \frac{23}{2!}(t-0)^2 + \frac{49}{3!}(t-0)^3 + \frac{167}{4!}(t-0)^4 + \cdots$$

$$y(t) = 7 + t + \frac{23}{2}t^2 + \frac{49}{6}t^5 + \frac{167}{24}t^2 + \cdots$$

**b)** Find a series solution for $t\,y'' + t^3 y' - 3y = 0$ such that $y(1) = 0$ and $y'(1) = 2$.

Solve for $y''$ and differentiate

$$y'' = -t^2 y' + 3t^{-1} y \quad \text{and then}$$
$$y''' = -t^2 y'' - \left(2t - 3t^{-1}\right) y' - 3t^{-2} y$$
$$y^{iv} = -t^2 y''' - \left(4t - 3t^{-1}\right) y'' - \left(2 + 6t^{-2}\right) y' + 6t^{-3} y$$




Evaluate the derivatives at $t = 1$

$$y''(1) = -2$$
$$y'''(1) = 4$$
$$y^{iv}(1) = -18$$

The solution is then

$$y(t) = 0 + 2(t-1) - \frac{2}{2}(t-1)^2 + \frac{4}{6}(t-1)^3 - \frac{1}{2}\cdot\frac{8}{4}(t-1)^4 + \cdots$$
$$y(t) = 2(t-1) - (t-1)^2 + \frac{2}{3}(t-1)^3 - \frac{3}{4}(t-1)^4 + \cdots$$

**a) Find a power series solution to**

$$y'' - 2y' - 3y = 0$$

$$y = \sum_{n=0}^{\infty} a_n x^n \qquad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 2\sum_{n=1}^{\infty} n a_n x^{n-1} - 3\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=1}^{\infty} 2n a_n x^{n-1} - \sum_{n=0}^{\infty} 3a_n x^n = 0$$

$$\underbrace{\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}}_{\substack{k=n-2 \\ n=k+2}} - \underbrace{\sum_{n=1}^{\infty} 2n a_n x^{n-1}}_{\substack{k=n-1 \\ n=k+1}} - \sum_{n=0}^{\infty} 3a_n x^n = 0$$

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k - \sum_{k=0}^{\infty} 2(k+1) a_{k+1} x^k - \sum_{k=0}^{\infty} 3a_k x^k = 0$$

$$\sum_{k=0}^{\infty} \left[ (k+2)(k+1) a_{k+2} - 2(k+1) a_{k+1} - 3a_k \right] x^k = 0 \quad \text{which means}$$

$$(k+2)(k+1) a_{k+2} - 2(k+1) a_{k+1} - 3a_k = 0$$

$$a_{k+2} = \frac{2(k+1) a_{k+1} + 3a_k}{(k+2)(k+1)}$$

$$a_{k+2} = \frac{2(k+1)a_{k+1} + 3a_k}{(k+2)(k+1)}$$

$$a_2 = \frac{2a_1 + 3a_0}{2 \cdot 1} = a_1 + \frac{3}{2}a_0$$

$$a_3 = \frac{4a_2 + 3a_1}{3 \cdot 2} = \frac{4a_2}{6} + \frac{3a_1}{6} = \frac{2}{3}\left(a_1 + \frac{3}{2}a_0\right) + \frac{1}{2}a_1 = \frac{2}{3}a_1 + a_0 + \frac{1}{2}a_1 = \frac{7}{6}a_1 + a_0$$

$$a_4 = \frac{6a_3 + 3a_2}{4 \cdot 3} = \frac{5}{6}a_1 + \frac{7}{8}a_0$$

$$a_5 = \frac{8a_4 + 3a_3}{5 \cdot 4} = \frac{41}{60}a_1 + \frac{13}{20}a_0$$

$$a_6 = \frac{14}{45}a_1 + \frac{73}{240}a_0$$

$$a_7 = \frac{347}{2520}a_1 + \frac{2}{15}a_0$$

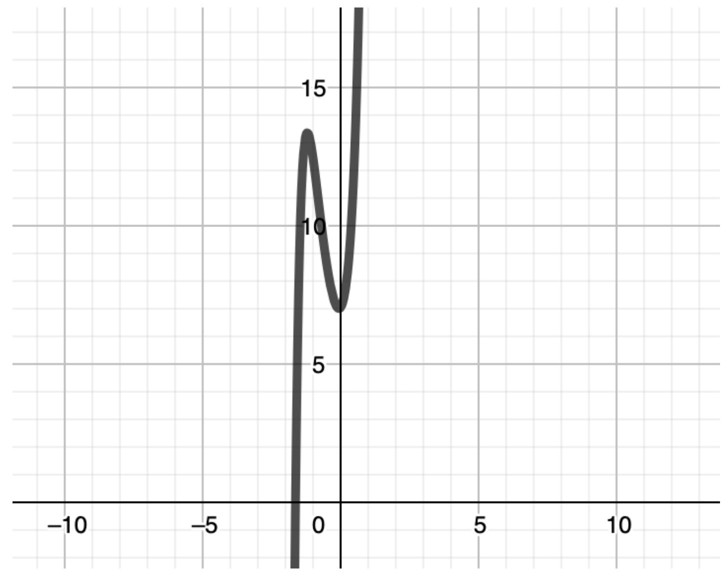
$$a_8 = \frac{103}{2016}a_1 + \frac{667}{13440}a_0$$

$$a_9 = \frac{443}{2592}a_1 + \frac{1003}{60480}a_0$$

Use initial conditions for $a_0 = y(0) = 7$ and $a_1 = y'(0) = 1$

$$a_0 = 7$$
$$a_1 = 1$$
$$a_2 = \frac{23}{2}$$
$$a_3 = \frac{49}{6}$$
$$a_4 = \frac{167}{24}$$
$$a_5 = \frac{157}{30}$$
$$a_6 = \frac{1757}{720}$$
$$a_7 = \frac{2699}{2520}$$
$$a_8 = \frac{16067}{40320}$$
$$a_9 = \frac{7439}{25920}$$

$$y_1(x) = 7 + x + \frac{23}{2}x^2 + \frac{49}{6}x^3 + \frac{167}{24}x^4 + \frac{157}{30}x^5 + \frac{1757}{720}x^6 + \frac{2699}{2520}x^7 + \frac{16067}{40320}x^8 + \frac{7439}{25920}x^9 + \cdots$$



**b) Find a power series solution to Airy's Equation**

$$y(x)=c_1\sqrt{x}\,J_{\frac{1}{3}}\left(\frac{2}{3}ix^{\frac{3}{2}}\right)+c_2\sqrt{x}\,J_{-\frac{1}{3}}\left(\frac{2}{3}ix^{\frac{3}{2}}\right)$$

$$y(x)=AiryAi[x]\,c_1+AiryBi[x]\,c_2$$

$$y''-xy=0$$

$$y=\sum_{n=0}^{\infty}a_n x^n \qquad y'=\sum_{n=1}^{\infty}n a_n x^{n-1} \qquad y''=\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}-x\sum_{n=0}^{\infty}a_n x^n=0$$

$$\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}-\sum_{n=0}^{\infty}a_n x^{n+1}=0$$

$$\underbrace{\sum_{n=2}^{\infty}n(n-1)a_n x^{n-2}}_{\substack{k=n-2\\ n=k+2}}-\underbrace{\sum_{n=0}^{\infty}a_n x^{n+1}}_{\substack{k=n+1\\ n=k-1}}=0$$

$$\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k-\sum_{k=1}^{\infty}a_{k-1}x^k=0$$

$$2a_2+\sum_{k=1}^{\infty}(k+2)(k+1)a_{k+2}x^k-\sum_{k=1}^{\infty}a_{k-1}x^k=0$$

$$2a_2+\sum_{k=1}^{\infty}\left[(k+2)(k+1)a_{k+2}-a_{k-1}\right]x^k=0$$ which means

$$2a_2=0 \qquad a_2=0$$

$$(k+2)(k+1)a_{k+2}-a_{k-1}=0$$

$$a_{k+2}=\frac{a_{k-1}}{(k+2)(k+1)}$$

$$2a_2 = 0$$
$$a_2 = 0$$

$$(k+2)(k+1)a_{k+2} - a_{k-1} = 0$$
$$a_{k+2} = \frac{a_{k-1}}{(k+2)(k+1)}$$

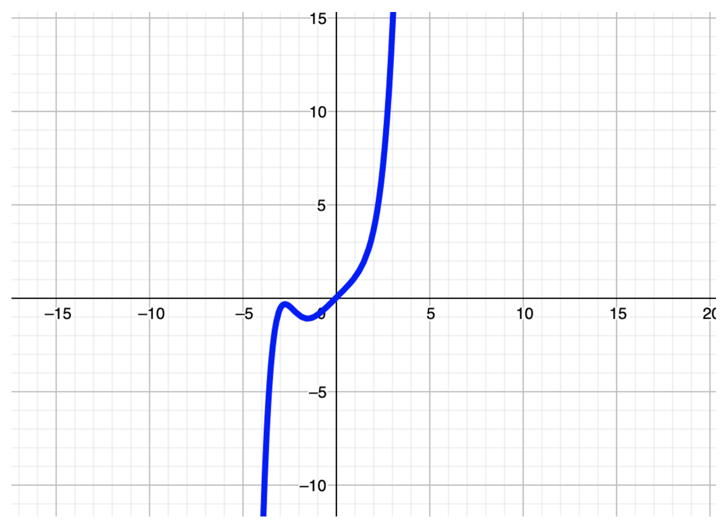
$$a_2 = 0$$
$$a_3 = \frac{a_0}{2\cdot 3}$$
$$a_4 = \frac{a_1}{3\cdot 4}$$
$$a_5 = \frac{a_2}{4\cdot 5} = 0$$
$$a_6 = \frac{a_5}{5\cdot 6} = \frac{a_0}{2\cdot 3\cdot 5\cdot 6}$$
$$a_7 = \frac{a_4}{6\cdot 7} = \frac{a_1}{3\cdot 4\cdot 6\cdot 7}$$
$$a_8 = \frac{a_5}{7\cdot 8} = 0$$
$$a_9 = \frac{a_6}{8\cdot 9} = \frac{a_6}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9}$$

$$a_2 = 0$$
$$a_3 = \frac{1}{6}a_0$$
$$a_4 = \frac{1}{12}a_1$$
$$a_5 = 0$$
$$a_6 = \frac{1}{180}a_0$$
$$a_7 = \frac{1}{504}a_1$$
$$a_8 = 0$$
$$a_9 = \frac{1}{12960}a_0$$

$a_0 = 1,\ a_1 = 0$ $a_0 = 1,\ a_1 = 0$

$$y_1(x) = 1 + \frac{1}{6}x^3 + \frac{1}{180}x^6 + \frac{1}{12960}x^9 + \cdots$$
$$y_2(x) = x + \frac{1}{12}x^4 + \frac{1}{504}x^7 + \cdots$$

$$y_1(x) = a_0\left[1 + \frac{1}{6}x^3 + \frac{1}{180}x^6 + \frac{1}{12960}x^9 + \cdots\right]$$
$$y_2(x) = a_1\left[x + \frac{1}{12}x^4 + \frac{1}{504}x^7 + \cdots\right]$$



Find the first few terms of the power series solution around $x_0 = 0$ of the given differential equation. If possible, determine the general series expansion and a closed form expression of the solution.

1. $y' = x$
2. $y' = x^2 + 2x + 1$
3. $y' = y + 1$
4. $y' - 2xy = 0$
5. $y' + (x + 2)y = 0$
6. $y'' - y = 0$
7. $y'' + xy = 0$
8. $y'' - x^2y = 0$
9. $y'' - xy' + 2y = 0$
10. $(1 + x^2)y'' - y' + y = 0$
11. $(1 + x^2)y'' - 4xy' + 6y = 0$