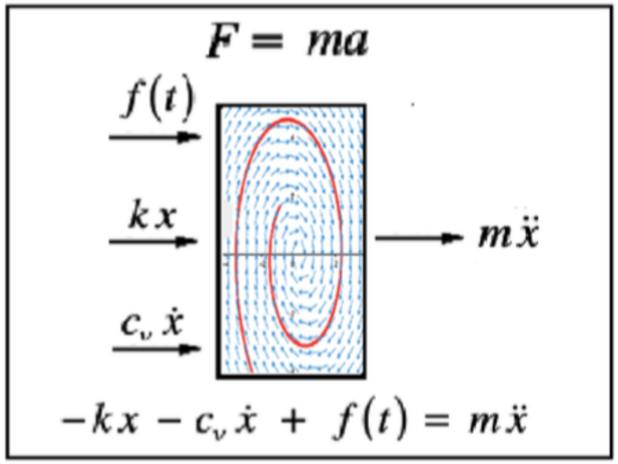


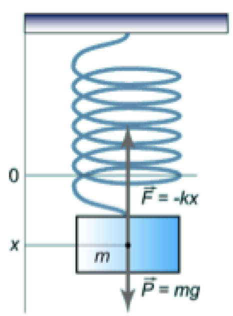
$$A\frac{d^2y}{dt^2} + B\frac{dy}{dt} + Cy = f(t)$$

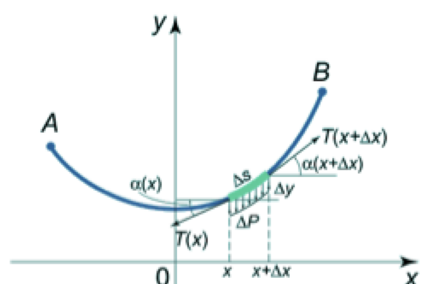
acceleration
curve bending

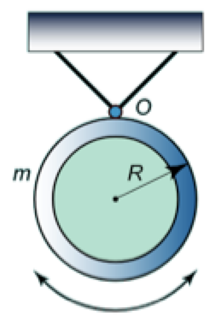
damping
friction
resistance

restoring force

"forcing" function







mth_211

$$Ay'' + By' + Cy = f(t)$$

constant coefficients

"forcing" function

Forced response to the exponential :

$$A\,y'' + B\,y' + C\,y = e^{st}$$

$$A\,y'' + B\,y' + C\,y = e^{st}$$

Try Ke^{st}

$$AK\,s^2 e^{st} + BK\,s e^{st} + CK\,e^{st} = e^{st}$$

$$K\left(As^2 + Bs + C\right) = 1$$

$$y_p(t) = \left(\frac{1}{As^2 + Bs + C}\right)e^{st}$$

$$Ay'' + By' + Cy = f(t)$$

constant coefficients

"forcing" function

Forced response to oscillatory input (undamped motion) :

$$A\,y'' + \qquad C\,y = \cos\beta t$$

Try $M\cos\beta t$

$$y_p(t) = \left(\frac{1}{C - \beta^2 A}\right)\cos\beta t$$

$$Ay'' + By' + Cy = f(t)$$

"forcing" function

constant coefficients

Forced response to oscillatory input :

$$A\,y'' + B\,y' + C\,y = \cos\beta t$$

$A\,y'' + B\,y' + C\,y = \cos\beta t$

Try $M\cos\beta t + N\sin\beta t$

$$y_p(t) = \left(\frac{1}{A^2\beta^4 + \beta^2 B^2 - 2A\beta^2 C + C^2}\right)\left(\beta B\sin\beta t - A\beta^2\cos\beta t\right)$$

Method of Undetermined Coefficients

$$Ay'' + By' + Cy = f(t)$$

constant coefficients

"forcing" function

$$y'' - 3y' + 2y = 10\sin 2t$$

$$y_n(t) = c_1 e^t + c_2 e^{3t} \qquad y_p(t) = \frac{3}{2}\cos 2t - \frac{1}{2}\sin 2t$$

$$y(t) = y_n(t) + y_p(t) = c_1 e^t + c_2 e^{3t} + \frac{3}{2}\cos 2t - \frac{1}{2}\sin 2t$$

Method of Undetermined Coefficients

$$Ay'' + By' + Cy = f(t)$$

constant coefficients

"forcing" function

$$y'' - 4y' + 2y = 2x^2$$

$$y_p(x) = x^2 + 4x + 7$$

Method of Undetermined Coefficients

$$Ay'' + By' + Cy = f(t)$$

constant coefficients

"forcing" function

$$y'' - 3y' + 2y = 5e^{2t}$$

Method of Undetermined Coefficients

$$Ay'' + By' + Cy = f(t)$$

constant coefficients

"forcing" function

$f(t)$	$y_p(t)$ guess
$ae^{\beta t}$	$Ae^{\beta t}$
$a\cos(\beta t)$	$A\cos(\beta t) + B\sin(\beta t)$
$b\sin(\beta t)$	$A\cos(\beta t) + B\sin(\beta t)$
$a\cos(\beta t) + b\sin(\beta t)$	$A\cos(\beta t) + B\sin(\beta t)$
n^{th} degree polynomial	$A_n t^n + A_{n-1}t^{n-1} + \cdots A_1 t + A_0$

Method of Undetermined Coefficients

$$Ay'' + By' + Cy = f(t)$$

constant coefficients

"forcing" function

$$y'' - 4y' + 12y = te^{4t}$$

$$y_p(t) = -\frac{1}{36}(3t+1)e^{4t}$$

Forcing Functions

Find a particular solution to the following -

a. $y'' - 3y' - 10y = \left(72x^2 - 1\right) e^{2x}$

b. $y'' - 3y' - 10y = 4xe^{6x}$

c. $y'' - 10y' + 25y = 6e^{-5x}$

d. $y'' - 10y' + 25y = 6e^{5x}$

e. $y'' + 4y' + 5y = 24\sin(3x)$

f. $y'' + 4y' + 5y = 8e^{-3x}$

g. $y'' - 4y' + 5y = e^{2x}\sin(x)$

h. $y'' - 4y' + 5y = e^{-x}\sin(x)$

i. $y'' - 4y' + 5y = 39xe^{-x}\sin(x) + 47e^{-x}\sin(x)$

j. $y'' - 4y' + 5y = 100$

k. $y'' - 4y' + 5y = e^{-x}$

l. $y'' - 4y' + 5y = 10x^2 + 4x + 8$

m. $y'' + 9y = e^{2x}\sin(x)$

Method of Variation of Parameters

$$Ay'' + By' + Cy = f(t)$$

constant coefficients

"forcing" function

Given $y_n(t) = c_1 y_1(t) + c_2 y_2(t)$

See section 3 - 10 of Dawkins

$$y_p(t) = -y_1 \int \frac{y_2 f(t)}{W(y_1, y_2)} dt + y_2 \int \frac{y_1 f(t)}{W(y_1, y_2)} dt$$

Method of Variation of Parameters

Given $y_n(t) = c_1\, y_1(t) + c_2\, y_2(t)$

See section 3 - 10 of Dawkins

$$y_p(t) = -y_1 \int \frac{y_2\, f(t)}{W(y_1, y_2)}\, dt + y_2 \int \frac{y_1\, f(t)}{W(y_1, y_2)}\, dt$$

$$y'' + 16\,y = 16\,t^2$$

$$y_p(t) = t^2 - \frac{1}{8}$$

Method of Variation of Parameters

Given $y_n(t) = c_1 y_1(t) + c_2 y_2(t)$

$$y_p(t) = -y_1 \int \frac{y_2 f(t)}{W(y_1, y_2)} dt + y_2 \int \frac{y_1 f(t)}{W(y_1, y_2)} dt$$

$$y'' - 10y' + 25y = 6e^{5t}$$

$$y_p(t) = 3t^2 e^{5t}$$

Forcing Functions

Find a particular solution to the following -

a. $y'' + 2y' + y = 2t^2 e^{-t}$

b. $y'' + 6y' + 9y = \dfrac{e^{-3t}}{t}$

c. $y'' + y = \sec t$

d. $y'' + y = \tan t$

e. $y'' + 2y' + y = e^{-x} \ln x$

Forcing Functions

$$y'' + y = \sin^2 t$$

Variation of Parameters:

$$y_p(t) = -\cos t \int \frac{\sin t \, \sin^2 t}{W(\cos t,\ \sin t)} dt + \sin t \int \frac{\sin t \, \sin^2 t}{W(\cos t,\ \sin t)} dt$$

$$y_p(t) = -\cos t \int \sin^3 t \, dt + \sin t \int \cos t \, \sin^2 t \, dt$$

$$y_p(t) = \frac{5}{6}\cos^2 t - \frac{1}{6}\cos 2t \, \cos^2 t + \frac{1}{3}\sin^4 t$$

$$y_p(t) = \frac{1}{6}\cos 2t + \frac{1}{2}$$

Undetermined Coefficients:

$$\sin^2 t = \frac{1}{2}\cos 2t - \frac{1}{2}$$

$$y_p(t) = A\cos 2t + B\sin 2t + C$$

$$y_p'(t) = -2A\sin 2t + 2B\cos 2t$$

$$y_p''(t) = -4A\cos 2t - 4B\sin 2t$$

$$-4A\cos 2t - 4B\sin 2t + A\cos 2t + B\sin 2t + C = \frac{1}{2}\cos 2t - \frac{1}{2}$$

$$-3A\cos 2t - 3B\sin 2t + B\sin 2t + C = \frac{1}{2}\cos 2t - \frac{1}{2}$$

$$A = -\frac{1}{6} \quad C = -\frac{1}{2}$$

$$y_p(t) = -\frac{1}{6}\cos 2t - \frac{1}{2}$$