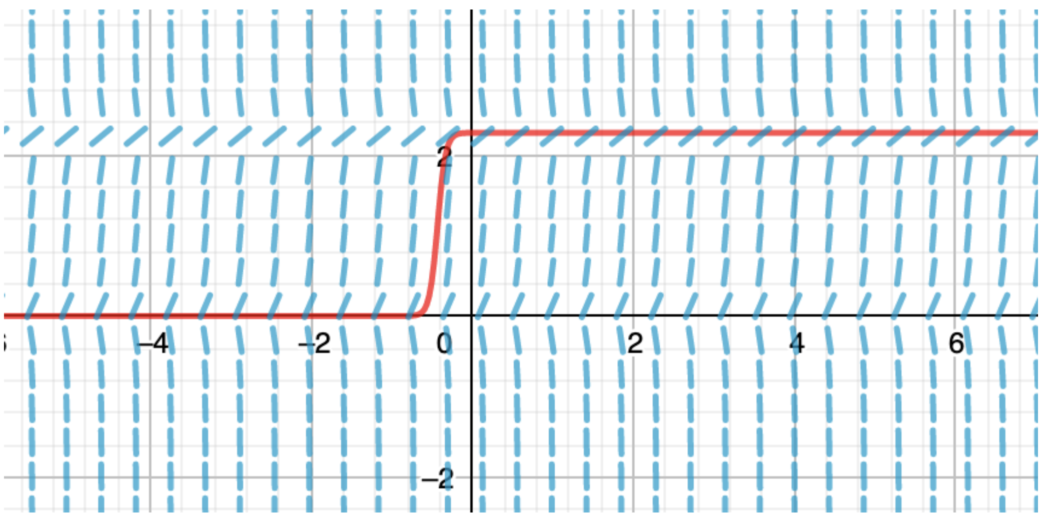
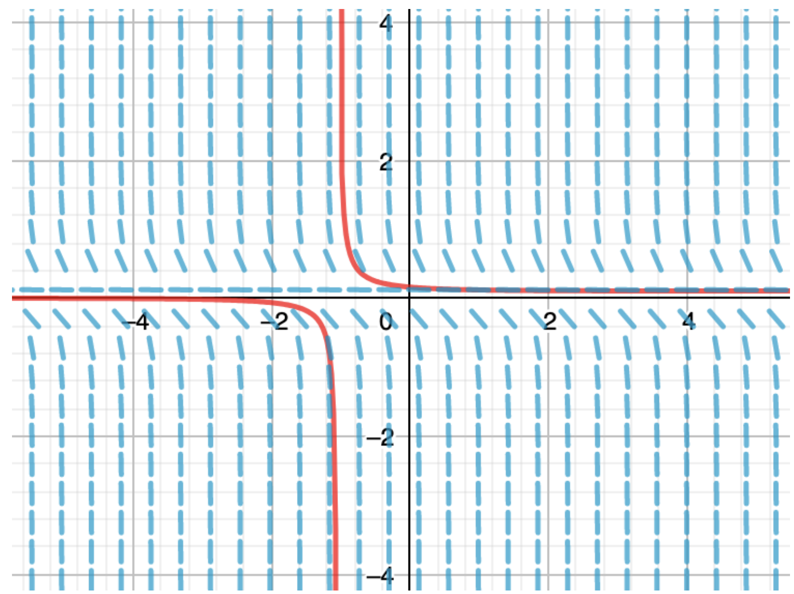
Growth and Decay
(Logistic Equation)

$$\frac{dy}{dt} = ay - by^2$$





$$\text{Let } z = \frac{1}{y} \quad \therefore \; \frac{dz}{dt} = -\frac{1}{y^2}\frac{dy}{dt} = -\frac{1}{y^2}\left(ay - by^2\right) = -az + b$$

$$\frac{dz}{dt} = -az + b \quad \Rightarrow \quad z = Ce^{-at} + \frac{b}{a} \text{ or } y = \frac{1}{Ce^{-at} + \frac{b}{a}}$$

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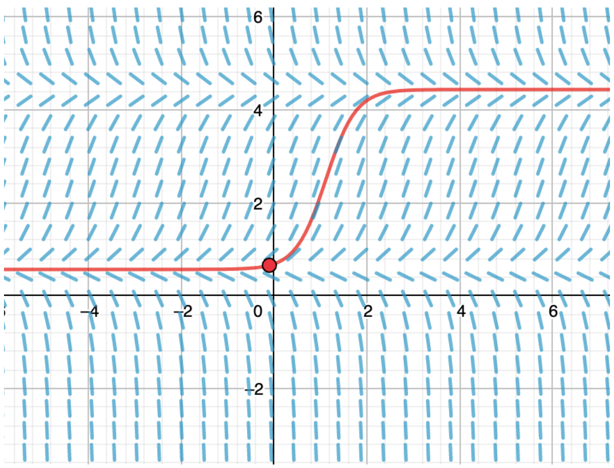
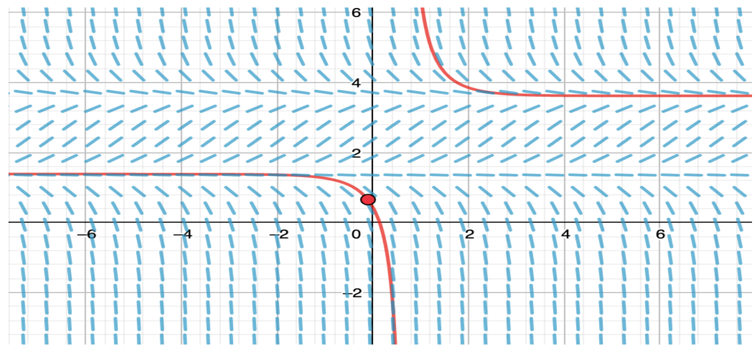
Growth and Decay
(Harvesting Equation)

$$\frac{dy}{dt} = ay - by^2 - h$$

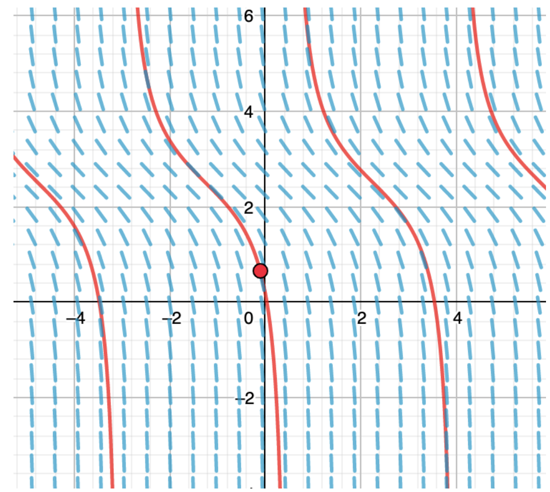
underharvesting : $h < a$



critical harvesting : $h = a$



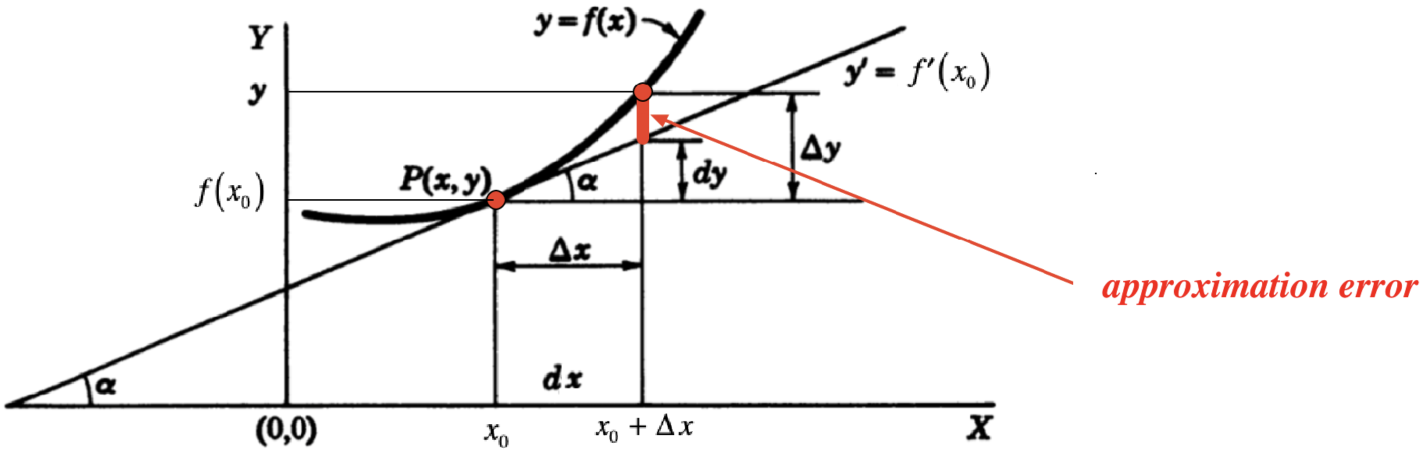
overharvesting : $h > a$



$$y(t) = \frac{1}{2b}\left(\left(a + \sqrt{4bh - a^2}\right)\tan\left(\frac{1}{2}(C - x)\sqrt{4bh - a^2}\right)\right)$$

Exact Equations

$$y - f(x_0) = f'(x_0)(x - x_0)$$

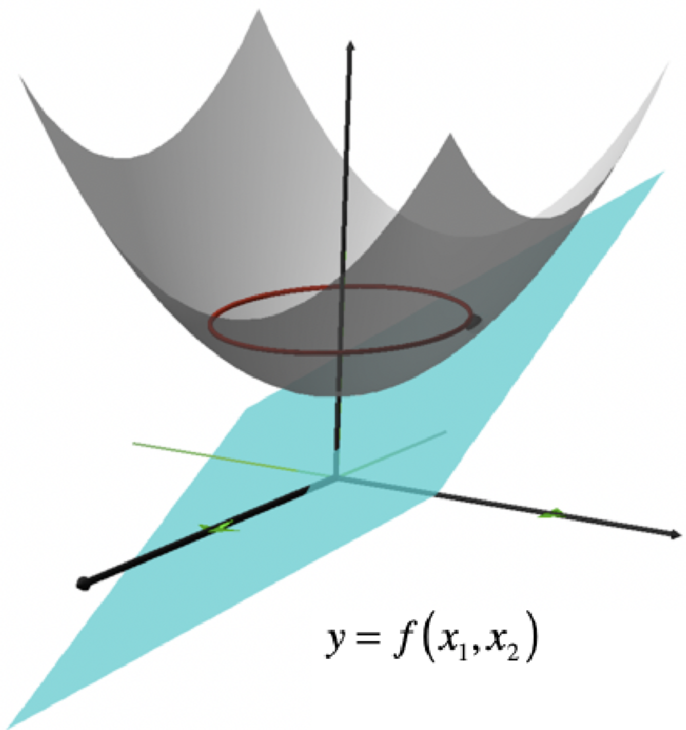


$$\frac{dy}{dx} = \frac{y - f(x_0)}{x - x_0} = f'(x_0) \text{ or}$$

$$y - f(x_0) = f'(x_0)(x - x_0) = f'(x_0)dx$$

$$dy = f'(x_0)dx$$

Exact Equations



$$y - f(x_{1_0}, x_{2_0}) = f_1'(x_{1_0}, x_{2_0})(x_1 - x_{1_0}) + f_2'(x_{1_0}, x_{2_0})(x_2 - x_{2_0})$$

$$dy = f_1'(x_{1_0}, x_{2_0})dx_1 + f_2'(x_{1_0}, x_{2_0})dx_2$$

Exact Equations

Example: $z = f(x, y) = 3x^2y + 5xy + y^3 + 5$ then the differential of z is

$$dz = \frac{\partial f(x,y)}{\partial x}dx + \frac{\partial f(x,y)}{\partial y}dy = (6xy + 5y)dx + (3x^2 + 5x + 3y^2)dy$$

If therefore we started with the differential expression

$(6xy + 5y)dx + (3x^2 + 5x + 3y^2)dy$ we would know that it is the total differential of $f(x, y)$,

A differential expression $M(x,y)dx + N(x,y)dy$ is called an **exact differential** if it is the total differential of a function $f(x, y)$, that is

$$M(x,y) = \frac{\partial}{\partial x}f(x,y) \quad \text{and} \quad N(x,y) = \frac{\partial}{\partial y}f(x,y)$$

Setting $(6xy + 5y)dx + (3x^2 + 5x + 3y^2)dy$ equal to zero, we obtain the differential equation

$$(6xy + 5y)dx + (3x^2 + 5x + 3y^2)dy = 0$$

whose solution is

$$f(x, y) = 3x^2y + 5xy + y^3 = C$$

Exact Equations

$$M\,dx + N\,dy = dU \quad \text{with} \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$M = \frac{\partial U}{\partial x} \quad \text{and} \quad N = \frac{\partial U}{\partial y}$$

To find U:

If $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ then

$U = \int M\,\partial x + f(y)$ then substitute U in $\dfrac{\partial U}{\partial y} = N$ to resolve $f(y)$

Solving exact ODEs (algor

$Mdx + Ndy = 0$ when $\dfrac{\partial M}{\partial y} =$

1. $U = \int M\,\partial x$
2. **Find** $\dfrac{\partial U}{\partial y}$
3. **Set** $\dfrac{\partial U}{\partial y}$ **=** N **to resolve**
4. **Set** $U = C$

Exact Equations

$$M\,dx + N\,dy = dU \quad \text{with} \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Example:

$$2xy\,dx + \left(x^2 + \cos y\right)dy = 0$$

$$M = 2xy \qquad N = x^2 + \cos y$$

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$$

Therefore U exists such that

$$\frac{\partial U}{\partial x} = 2xy\ , \quad \frac{\partial U}{\partial y} = x^2 + \cos y$$

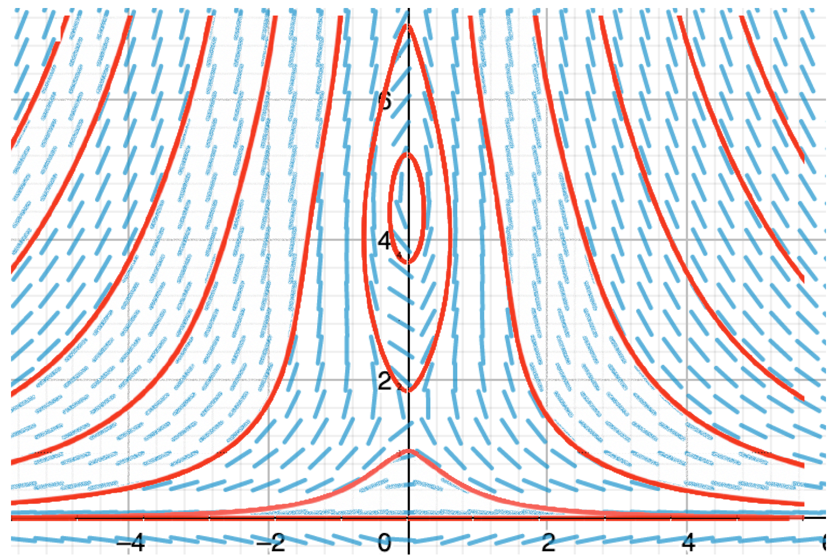
Integrating the first equation with respect to x gives $U = x^2y + f(y)$

Now, substituting into the second equation, we have

$$x^2 + f'(y) = x^2 + \cos y$$

$$\Rightarrow\ f'(y) = \cos y \quad \text{or} \quad f(y) = \sin y$$

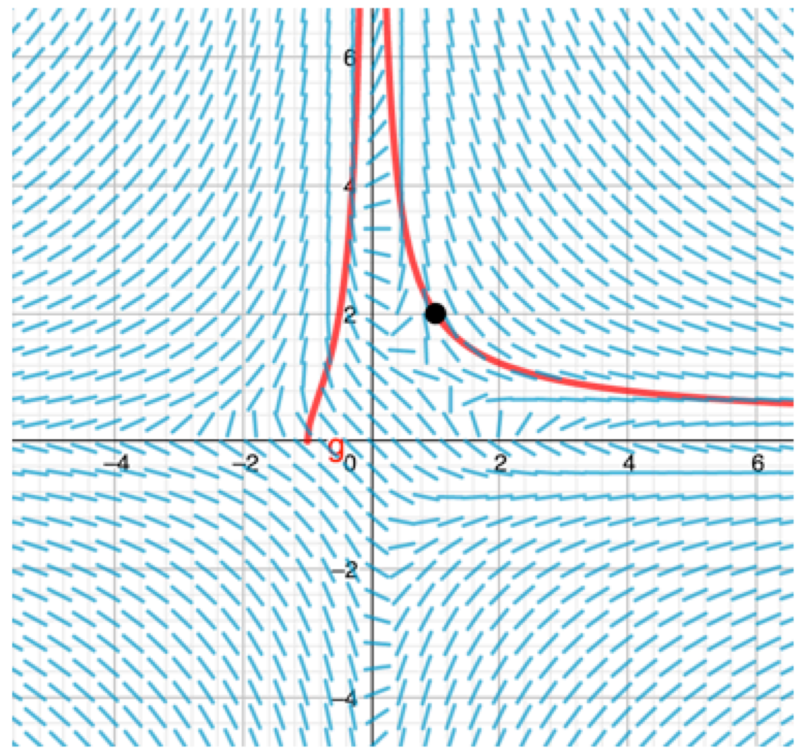
Hence $U = x^2y + \sin y$ and the solution is $x^2y + \sin y = C$



Exact Equations

$$M\,dx + N\,dy = dU \quad \text{with} \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

- Solve: $y' = \dfrac{xy^2 - 1}{1 - x^2 y}$ given $y(1) = 2$



$$\left(xy^2 - 1\right)dx + \left(x^2 y - 1\right)dy = 0,$$

$$M = xy^2 - 1 \qquad N = x^2 y - 1$$

$$\frac{\partial}{\partial y}\left(xy^2 - 1\right) = 2xy = \frac{\partial}{\partial x}\left(x^2 y - 1\right)$$

$$\int \left(xy^2 - 1\right)\partial x = \frac{1}{2}x^2 y^2 - x + f(y)$$

$$\frac{\partial}{\partial y}\left(\frac{1}{2}x^2 y^2 - x + f(y)\right) = x^2 y + f'(y)$$

Now $x^2 y + f'(y) = x^2 y - 1$

$\therefore \displaystyle\int f'(y)dy = f(y) = -y$

So $U = \frac{1}{2}x^2 y^2 - x - y$ and $\frac{1}{2}x^2 y^2 - x - y = C$

$y(1) = 2 \Rightarrow C = -1$

Solving Exact Differential Equations

- **Find the general solution to each of the following exact differential equations.**

a. $2xy + y^2 + [2xy + x^2]\frac{dy}{dx} = 0$

b. $2xy^3 + 4x^3 + 3x^2y^2\frac{dy}{dx} = 0$

c. $2 - 2x + 3y^2\frac{dy}{dx} = 0$

d. $1 + 3x^2y^2 + [2x^3y + 6y]\frac{dy}{dx} = 0$

e. $4x^3y + [x^4 - y^4]\frac{dy}{dx} = 0$

f. $1 + \ln|xy| + \frac{x}{y}\frac{dy}{dx} = 0$

g. $1 + e^y + xe^y\frac{dy}{dx} = 0$

h. $e^y + [xe^y + 1]\frac{dy}{dx} = 0$

Solving Exact Differential Equations

f.

$$(1+\ln xy)dx + \left(\frac{x}{y}\right)dy = 0,$$

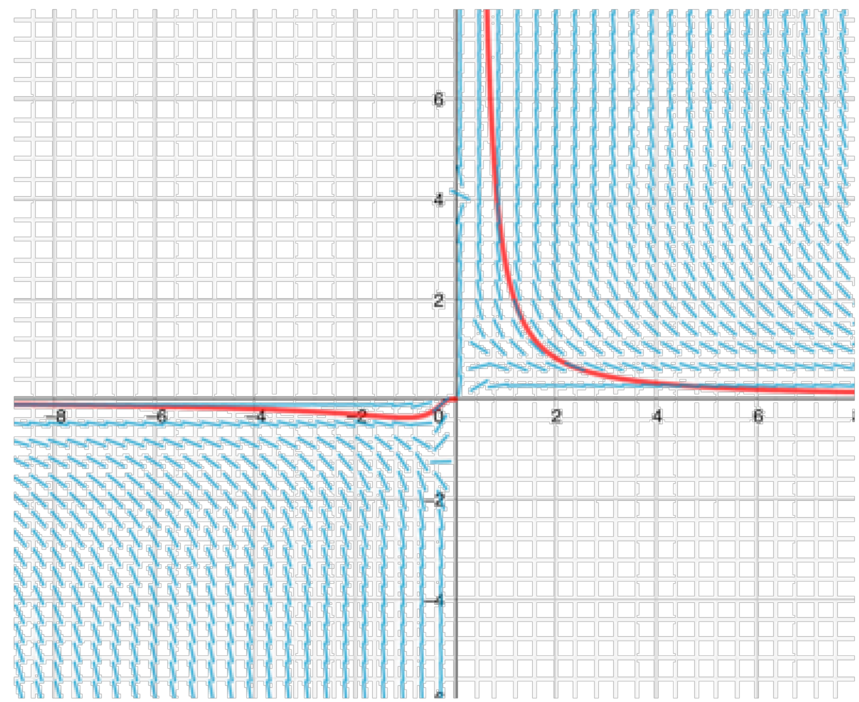
$$M = 1+\ln xy \qquad N = \frac{x}{y}$$

$$\frac{\partial}{\partial y}(1+\ln xy) = \frac{1}{y} = \frac{\partial}{\partial x}\left(\frac{x}{y}\right)$$

$$\int (1+\ln xy)\partial x = x\ln xy + f(y)$$

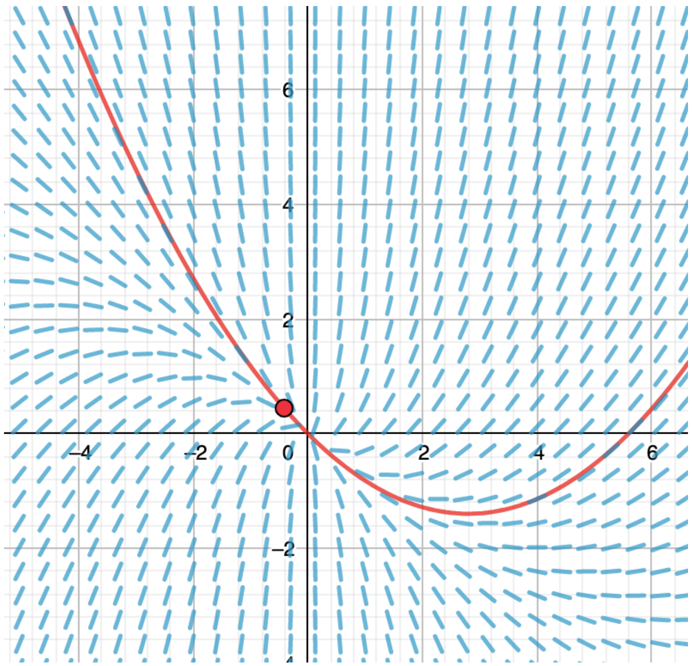
$$\frac{\partial}{\partial y}\left(x\ln xy + f(y)\right) = \frac{x}{y} + f'(y)$$

So the $\frac{x}{y} + f'(y) = \frac{x}{y}$



$$\therefore f'(y) \Rightarrow f(y) = 0 \quad \text{and} \quad x\ln xy = C$$

Miscellaneous Substitutions
(Homogeneous Equations)



$$u = \frac{y}{x} \longmapsto \frac{du}{dx}x + u = f(u)$$

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \qquad \text{Let } u = \frac{y}{x} \text{ or } y = ux$$

$$\text{then } \frac{dy}{dx} = \frac{du}{dx}x + u \quad \therefore \quad \frac{du}{dx}x + u = f(u)$$

Example:

$$\frac{dy}{dx} = \frac{x^2 + 2xy}{x^2}$$

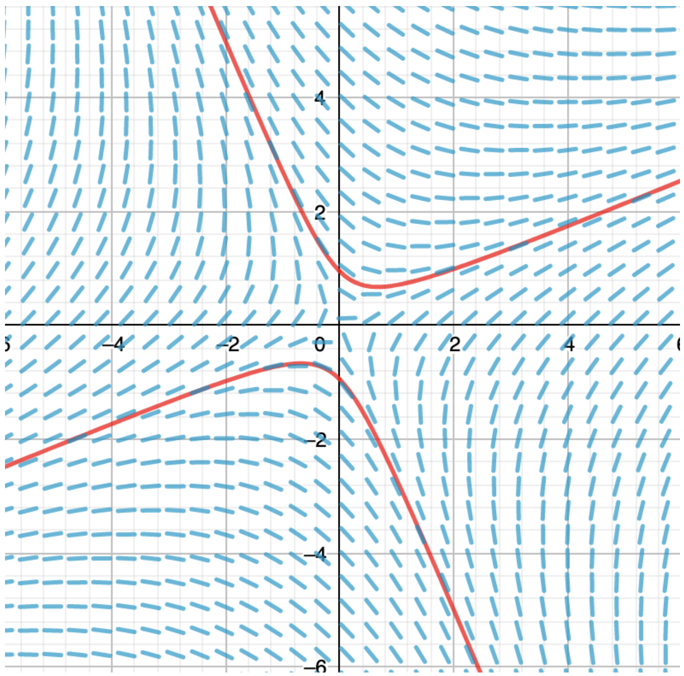
$$\frac{dy}{dx} = \frac{x^2 + 2xy}{x^2} = 1 + 2\frac{y}{x}$$

$$\text{so } \frac{du}{dx}x + u = f(u) = 1 + 2u$$

$$u = Cx - 1$$

$$y = Cx^2 - x$$

Miscellaneous Substitutions
(Homogeneous Equations)



$$\frac{dy}{dx} = \frac{x-y}{x+y} = \frac{1-\frac{y}{x}}{1+\frac{y}{x}}$$

$$\frac{dy}{dx} = \frac{x-y}{x+y} = \frac{1-\frac{y}{x}}{1+\frac{y}{x}}$$

so $\frac{du}{dx}x + u = f(u) = \frac{1-u}{1+u}$

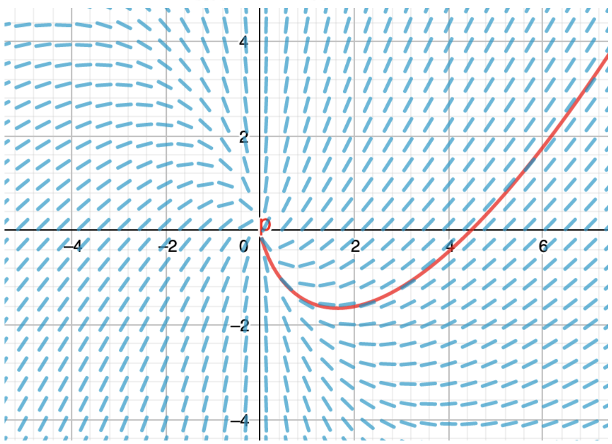
$$u = -1 \pm \sqrt{2 - \frac{C}{x^2}}$$

$$y = -x \pm x\sqrt{2 - \frac{C}{x^2}}$$

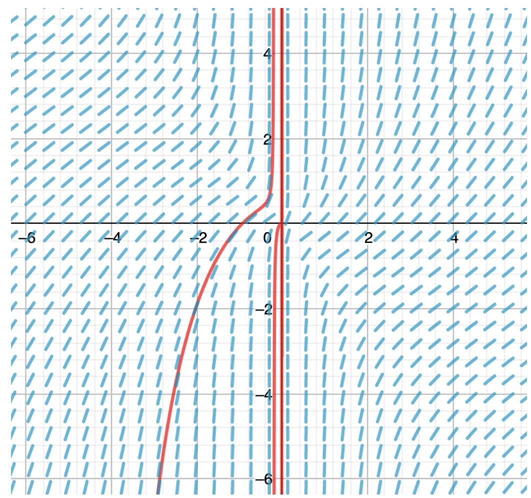
Miscellaneous Substitutions
(Homogeneous Equations)

Try the following:

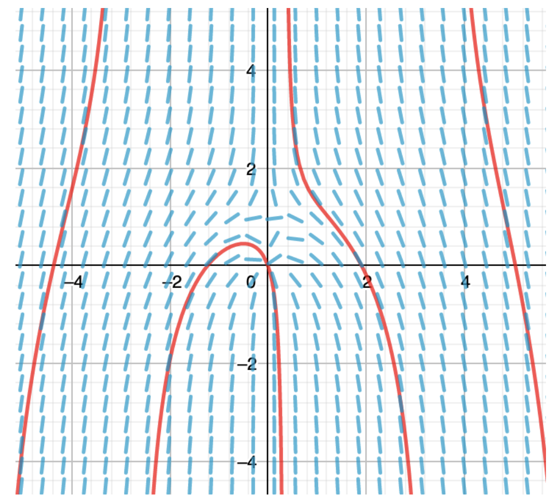
a) $\dfrac{dy}{dx} = \dfrac{x+y}{x}$



b) $\dfrac{dy}{dx} = \dfrac{x^2 + xy + y^2}{x^2}$



c) $\left(y - y^2 - x^2\right)dx - x\,dy = 0$



Solve the following homogeneous equations.

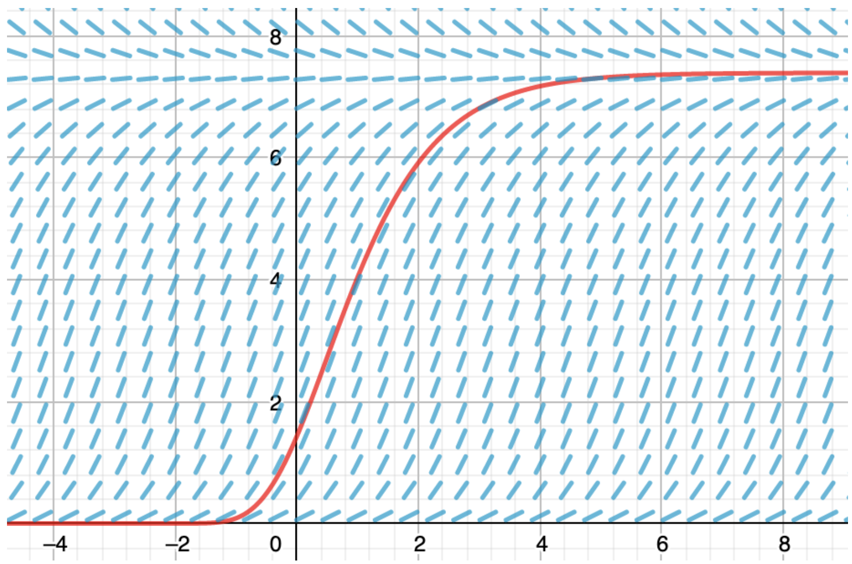
1. $\dfrac{dy}{dx} = 1 + \dfrac{y}{x}$

2. $\dfrac{dy}{dx} = \dfrac{y}{x} + \dfrac{y^2}{x^2};\quad y(1) = 1$

3. $xy' = 2x + 3y$

Gompertz Equation

$$\frac{dy}{dt} = y\left(a - \ln y\right)$$



Let $z = \ln y$, then $\frac{dz}{dt} = \frac{1}{y}\frac{dy}{dt}$

$$\therefore \frac{dz}{dt} = \frac{1}{y}\left(ay - y\ln y\right) = a - z$$

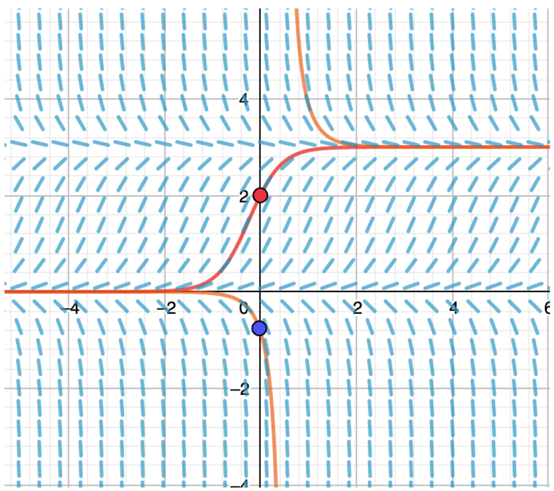
$$\Rightarrow y(t) = e^{a - e^{C-x}}$$

The function was originally designed to describe human mortality, but since has been modified to be applied in biology, with regard to detailing populations. This function is especially useful in describing the rapid growth of a certain population of organisms while also being able to account for the eventual horizontal asymptote, once the carrying capacity is determined (plateau cell/population number). In the 1960s A.K. Laird for the first time successfully used the Gompertz curve to fit data of growth of tumors. In fact, tumors are cellular populations growing in a confined space where the availability of nutrients is limited. A generalized logistic function, also called the Richards growth curve, is widely used in modelling COVID-19 infection trajectories. This curve reduces to the logistic function, and in many cases converges to the Gompertz function.

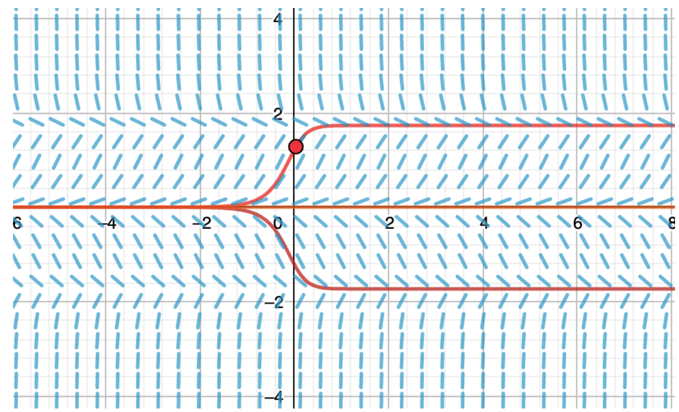
Bernoulli Equations

$$\frac{dy}{dx} + p(x)y = q(x)y^n$$

$n = 2$



$n = 3$ ("pitchfork" function)



Let $v = y^{1-n} = \frac{y}{y^n}$, $\frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{y^n}{1-n}\frac{dv}{dx}$$

$$\frac{y^n}{1-n}v' + p(x)vy^n = q(x)y^n$$

$$\frac{1}{1-n}v' + p(x)v = q(x)$$

Bernoulli Equations

$$y' + p(x)y = q(x)y^n$$

Solving Bernoulli ODEs (algorithm)

1. Divide by y^n
2. Set the $y-factor$ of $p(x)$ to v
3. Calculate v'
4. Rewrite in terms of v and v'
5. Solve for v
6. Substitute back y

Bernoulli Equations

$$y' + p(x)y = q(x)y^n$$

Solving Bernoulli ODEs (algorithm)

1. Divide by y^n
2. Set the y – *factor* of $p(x)$ to v
3. Calculate v'
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6. Substitute back y

Example:

Solve $y' + \left(\frac{4}{x}\right)y = x^3y^2$

$$y^{-2}y' + \left(\frac{4}{x}\right)y^{-1} = x^3$$

so $v = y^{-1}$ $v' = -y^{-2}y'$

$$-v' + \left(\frac{4}{x}\right)v = x^3$$

Solving we get $v = -x^4 \ln x + Cx^4$ or

$$y = \frac{-1}{x^4 \ln x + Cx^4}$$

Bernoulli Equations

$$\frac{dy}{dx} + p(x)y = q(x)y^n$$

Solve the following Bernoulli equations.

a. $t\dfrac{dy}{dt} + y = \dfrac{1}{y^2}$

b. $\dfrac{dy}{dt} - y = e^t y^2$

c. $\dfrac{dy}{dt} = y\left(xy^3 - 1\right)$

d. $t\dfrac{dy}{dt} - (1+t)\,y = ty^2$

e. $t^2\dfrac{dy}{dt} + y^2 = t\,y$

f. $3\left(1+t^2\right)\dfrac{dy}{dt} = 2ty\left(y^3 - 1\right)$

g. $t^2\dfrac{dy}{dt} - 2ty = 3y^4, \ y(1) = \dfrac{1}{2}$

h. $\sqrt{y}\dfrac{dy}{dt} + \sqrt{y^3} = 1, \ y(0) = 4$