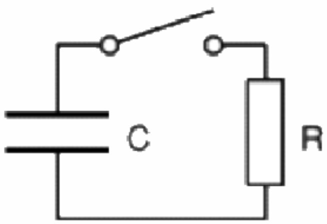
Growth and Decay

$$\frac{dy}{dt} = ay$$

Discharging a capacitor



Carbon 14 is a common form of carbon which decays over time. The amount of Carbon 14 contained in a preserved plant is modeled by the equation

$$\frac{dy}{dt} = -at \qquad \text{or} \qquad y(t) = y_0 e^{-at}$$

Time in this equation is measured in years from the moment when the plant dies ($t = 0$) and the amount of Carbon 14 remaining in the preserved plant is measured in micrograms (a microgram is one millionth of a gram). So when $t = 0$ the plant contains y_0 micrograms of Carbon 14.

a) The half-life of Carbon 14, that is the amount of time it takes for half of the Carbon 14 to decay, is approximately 5730 years. Use this information to find the constant a.

b) If there is currently 15% of the Carbon 14 remaining in the preserved plant, approximately when did the plant die ?

Growth and Decay

- An experimental population of fruit flies increases according to the law of exponential growth. There were 100 flies after the second da of the experiment and 300 flies after the fourth day. Approximately how many flies were in the original population?

- The number of bacteria in a culture is increasing according to the law of exponential growth. There are 125 bacteria in the culture after 2 hours and 350 bacteria after 4 hours.
  - (a) Find the initial population.
  - (b) Write an exponential growth model for the bacteria population. Let represent time in hours.
  - (c) Use the model to determine the number of bacteria after 8 hours.
  - (d) After how many hours will the bacteria count be 25,000?

- A certain radioactive material is known to decay at a rate proportional to the amount present. Initially, 100 grams of the substance are present, but after 50 years the mass decays to 75 grams. Find an expression for the mass of the material at any time. What is the half-life of the material?

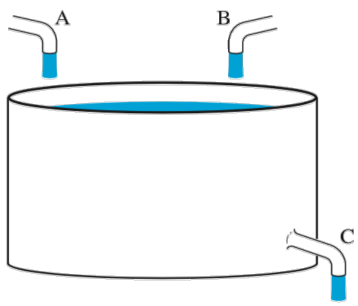
Growth and Decay

• A cypress beam found in the tomb of Sneferu in Egypt contained 55% of the radioactive carbon-14 that is found in living cypress wood. Estimate the age of the tomb. (half life of carbon 14 approx. 5730 years)

★The U.S. government has dumped roughly 100,000 barrels of radioactive waste into the Atlantic and Pacific oceans. The waste is mixed with concrete and encased in steel drums. The drums will eventually rust, and seawater will gradually leach the radioactive material from the concrete and diffuse it throughout the ocean. It is assumed that the leached radioactive material will be so diluted that no environmental damage will result. However, scientists have discovered that one of the pollutants, americium 241, is sticking to the ocean floor near the drums. Given that americium 241 has a half-life of 258 years, how long will it take for the americium 241 to be reduced to 5% of its present amount?

★In 1964, Soviet scientists made a new element with atomic number 104, called El04, by bombarding plutonium with neon ions. The half-life of this new element is 0.15 second, and it was produced at a rate of $2 \times 10–5$ micrograms per second. Assuming that none was present initially, how much El04 is present after t seconds?

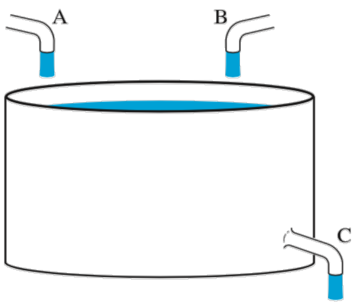
Mixing Models



Consider a large vat containing sugar water that is to be made into soft drinks. Suppose:

- The vat contains 100 gallons of liquid. Moreover, the amount flowing in is the same as the amount flowing out, so there are always 100 gallons in the vat.
- The vat is kept well mixed, so the sugar concentration is uniform throughout the vat.
- Sugar water containing 5 tablespoons of sugar per gallon enters the vat through pipe A at a rate of 2 gallons per minute.
- Sugar water containing 10 tablespoons of sugar per gallon enters the vat through pipe B at a rate of 1 gallon per minute.
- Sugar water leaves the vat through pipe C at a rate of 3 gallons per minute.

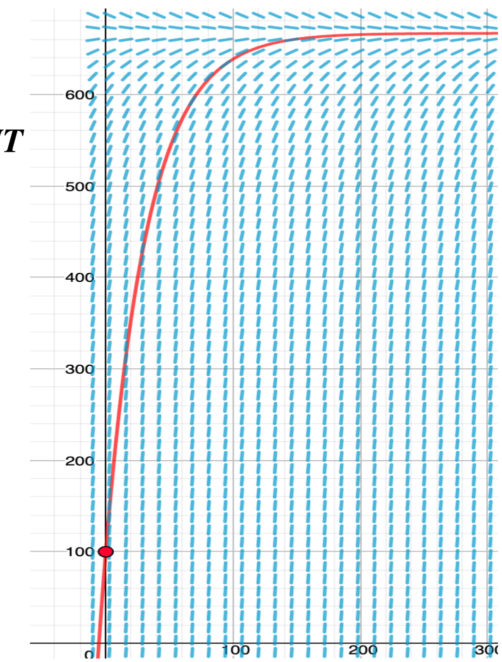
Mixing Models



RATE-IN →

$$\frac{dy}{dt} = RATE_{IN} - RATE_{OUT}$$

→ *RATE-OUT*



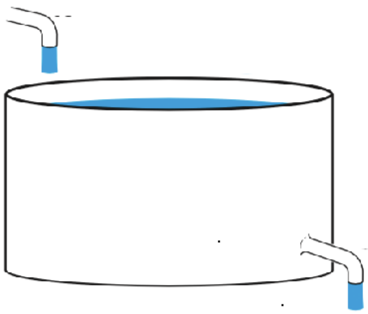
$$RATE_{IN} - RATE_{OUT}$$

$$\text{concentration}_{IN} \cdot \text{flow\_rate}_{IN} - \text{concentration}_{OUT} \cdot \text{flow\_rate}_{OUT}$$

$$\frac{dy}{dt} = \left(5\,\frac{T}{gal} \cdot 2\,\frac{gal}{\min} + 10\,\frac{T}{gal} \cdot 1\,\frac{gal}{\min}\right) - \left(\frac{y}{100}\right)\frac{T}{gal} \cdot 3\,\frac{gal}{\min}$$

$$\frac{dy}{dt} = 20 - \frac{3}{100}y$$

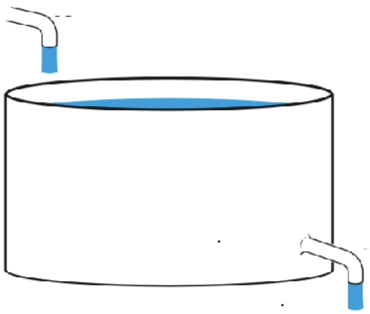
Mixing Models



A tank initially contains 50 gal of pure water. A salt solution containing 2 pounds of salt per gallon of water is poured into the tank at a rate of 3 gal/min. The mixture is stirred and is drained out of the tank at the same rate.

- **Find the initial-value problem that describes the amount y of salt in the tank at any time.**
- **Find the amount of salt in the tank at any time.**
- **Find the amount of salt in the tank after 20 minutes.**
- **Find the amount of salt in the tank after a "long time."**

Mixing Models



A tank initially contains 100 gallons of water in which 10 pounds of salt are dissolved. A salt solution containing 0.5 pound of salt per gallon is poured into the tank at a rate of 1 gal/min. The mixture in the tank is stirred and drained off at the rate of 2 gal/min.

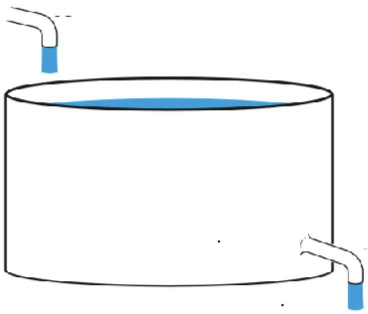
- Find the initial-value problem that describes the amount of salt $y(t)$ in the tank until the tank is empty.

- Find the amount of salt $y(t)$ in the tank until the tank is empty.

- Find the concentration $c(t)$ of salt in the tank until the tank is empty.

- Find the concentration of salt in the tank at the exact time the tank becomes empty.

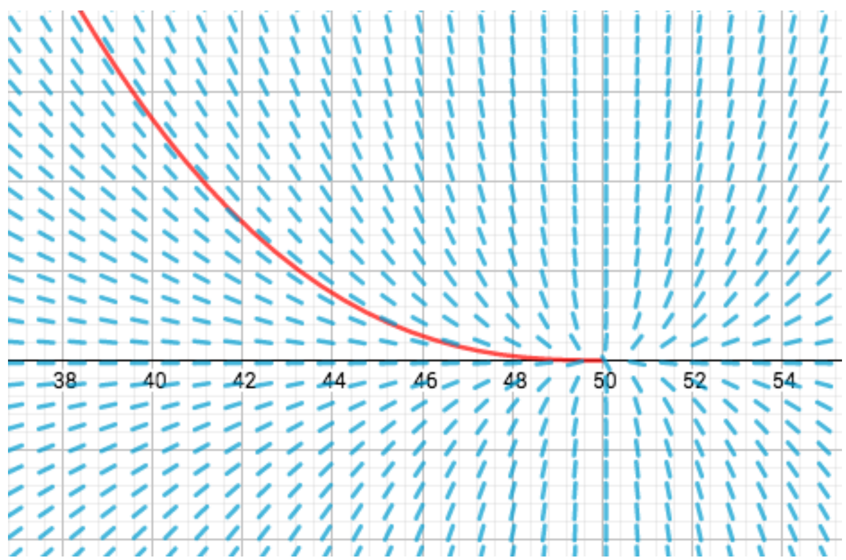
Mixing Models



A tank initially contains 100 gallons of brine whose salt concentration is 3 lb/gal. Fresh water is poured into the tank at a rate of 3 gal/min, and the well-stirred mixture flows out of the tank at the rate of 5 gal/min.

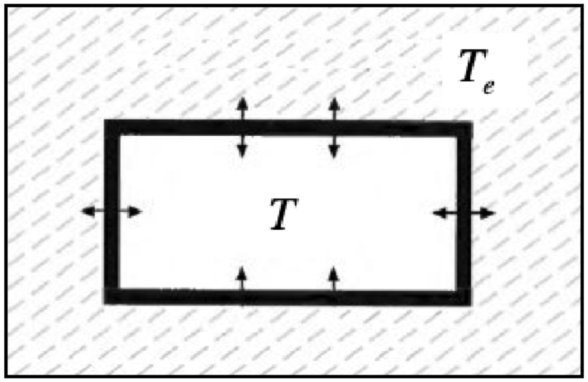
- **Find the initial-value problem that describes the amount of salt in the tank.**
- **Find the amount of salt in the tank after time t .**

$$\frac{dy}{dt} = \frac{-5y}{100-2t} \quad ; y(0) = 3$$



$$y(t) = \frac{3}{1000}(100-2t)^{\frac{5}{2}}$$

Conduction/Diffusion Models



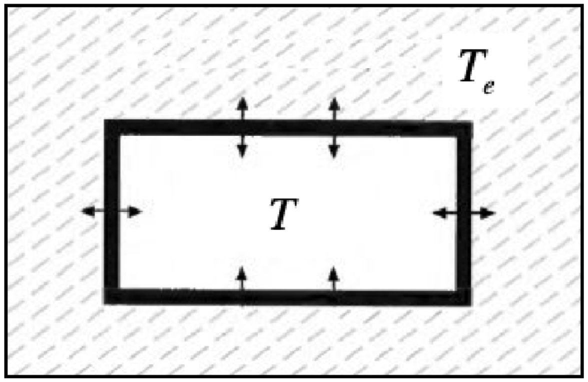
$$\frac{dT}{dt} \propto (T_e - T)$$

$$\frac{dT}{dt} = k(T_e - T)$$

At 12:00 midnight, with the temperature inside SLC at 70°F and the outside temperature at 20°F, the furnace breaks down. Two hours later the temperature in the building has fallen to 50°F.

- **Find the initial-value problem that describes the temperature inside the building for the remainder of the night. We assume that the outside temperature remains constant at 20°F.**
- **Determine the temperature in the building for the remainder of the night.**
- **Determine when the temperature in the building will fall to 40°F.**

Conduction/Diffusion Models



$$\frac{dT}{dt} \propto (T_e - T)$$

$$\frac{dT}{dt} = k(T_e - T)$$

$$\frac{dT}{dt} = k(20 - T); \; T(0) = 70^\circ F$$

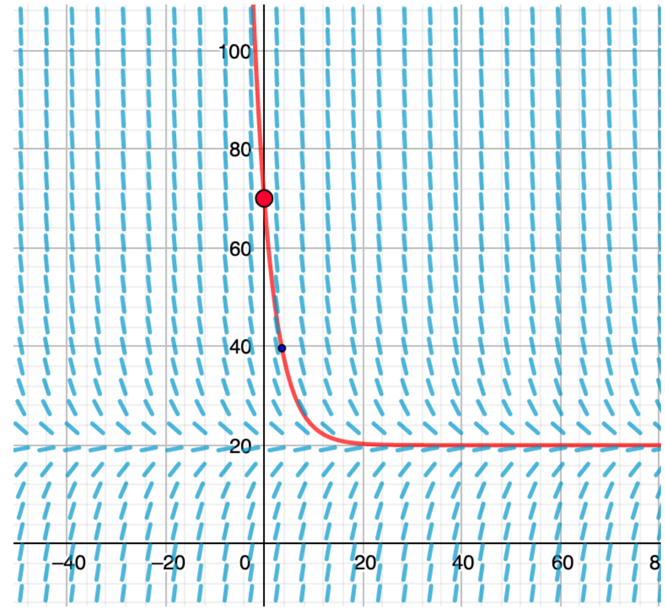
$$T(t) = 20 + 50e^{-kt}$$

To find k, we have $T(2) = 50$

$$T(t) = 20 + 50e^{-0.255t}$$

Solve for t,

$40 = 20 + 50e^{-0.255t}, \quad t \approx 3.592$ hr



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Kirchhoff's Law

The algebraic sum of all the voltage drops around an eletric loop or circuit is zero.

Voltage drops

$$RI = R\frac{dQ}{dt} \qquad L\frac{dI}{dt} \qquad \frac{Q}{C}$$

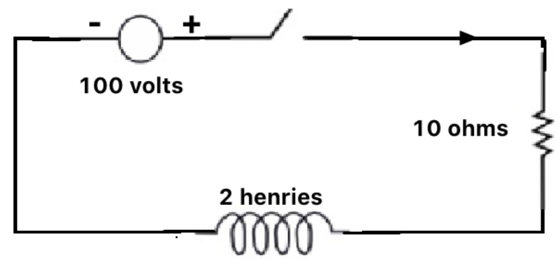
$$L\frac{dI}{dt} + RI = E$$

$$R\frac{dQ}{dt} + \frac{Q}{C} = E$$

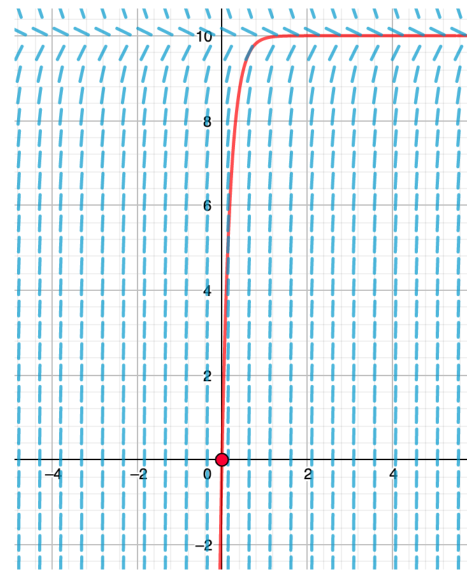
Kirchhoff's Law

The algebraic sum of all the voltage drops around an eletric loop or circuit is zero.

- **A generator having emf 100 volts is connected in series with a 10 ohm resistor and an inductor of 2 henries. If the switch is closed at time $t = 0$, set up a differential equation for the current and determine the current at time t.**

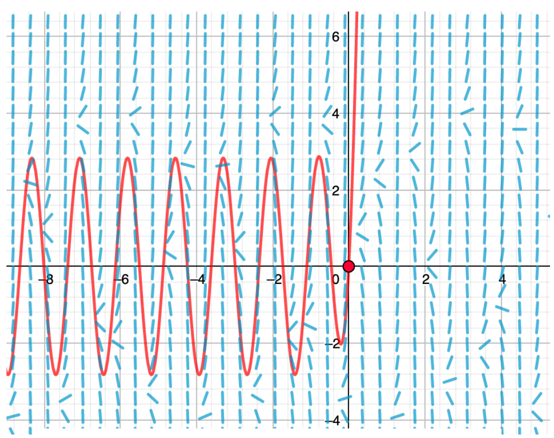


We have the voltage supplied at 100 volts, the voltage drop across the resistance (RI) = 10 volts, and the voltage drop across the inductor ($L\, dI/dt$) = 2 dI/dt.



$$I(t) = 10 + C e^{-5t}$$

- **Replace the 100 volt generator in the first example by one having an emf of 20 cos(5t) volts.**

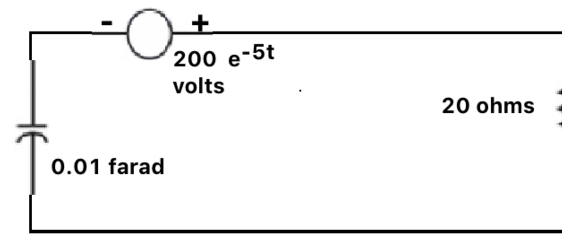


$$I(t) = C e^{-5t} + \cos 5t + \sin 5t$$

Kirchhoff's Law

The algebraic sum of all the voltage drops around an eletric loop or circuit is zero.

- **A decaying emf $E = 200e^{-5t}$ is connected in series with a 20 ohm resistor and a 0.01 farad capacitor. Assuming $Q = 0$ at $t = 0$, find the charge and the current at any time t. Show that the charge reaches a maximum, calculate it and find when it is reached.**



$$R\frac{dQ}{dt} + \frac{Q}{C} = E$$

$$20\,Q' + \frac{Q}{0.01} = 200\,e^{-5t}$$

$$Q' + 5Q = 10\,e^{-5t}$$

$$Q(t) = 10t\,e^{-5t} + Ce^{-5t}$$

$$\frac{dQ}{dt} = 0 \text{ when } t = \frac{1}{5} \quad \text{and} \quad Q\left(\frac{1}{5}\right) = \frac{2}{e} \approx$$