Separable Differential Equations
(Separation of Variables)

$$\frac{dy}{dx} = \frac{x + xy^2}{4y}$$

$$\frac{dy}{dx} = f(x)$$

$$\int \frac{dy}{dx} = \int f(x)dx$$

$$y(x) = \int f(x)dx + C$$

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

$$g(y)\frac{dy}{dx} = f(x)$$

$$\int g(y)\frac{dy}{dx}\,dx = \int f(x)dx$$

let $y = g(y)$, then $dy = \frac{dy(x)}{dx}dx$

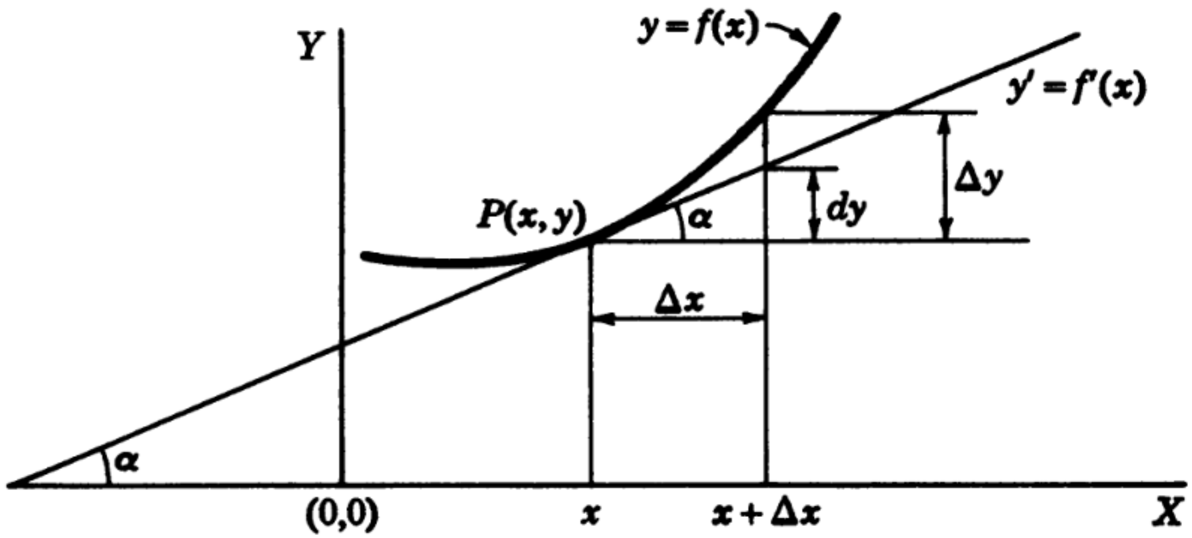
or $\int g(y)\,dy = \int f(x)dx + C$

mth_211

Separable Differential Equations
(Separation of Variables)

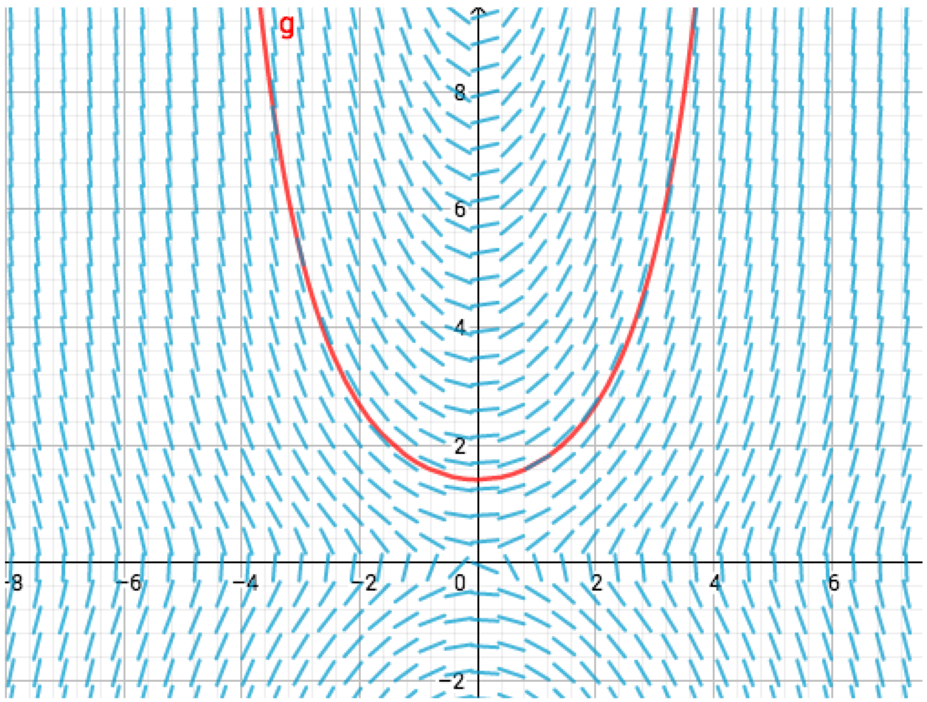
$$\frac{\Delta y}{\Delta x} \overset{?}{\propto} \frac{dy}{dx}$$

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{\lim_{\Delta x \to 0} \Delta y}{\lim_{\Delta x \to 0} \Delta x} \overset{?}{=} \frac{\lim_{\Delta y \to 0} \Delta y}{\lim_{\Delta x \to 0} \Delta x} = \frac{dy}{dx}$$



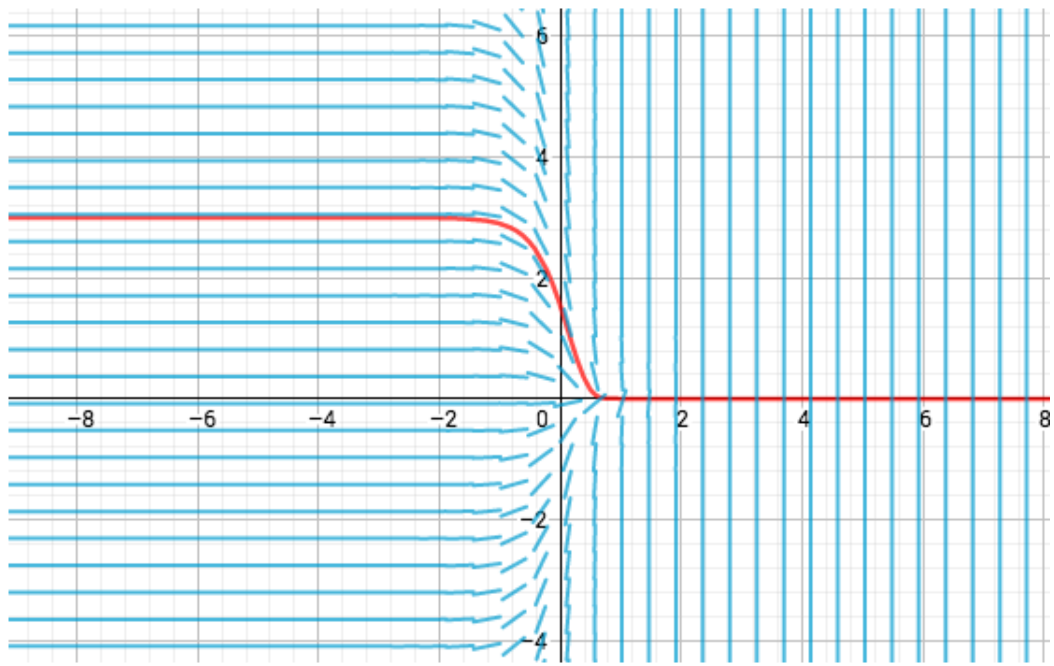
Separable Differential Equations
(Separation of Variables)

$$\frac{dy}{dx} = \frac{x + xy^2}{4y}$$



Separable Differential Equations (Separation of Variables)

$$2y\,dx + e^{-3x}\,dy = 0$$



- Use separation of variables to solve the following.

1. $\frac{dy}{dx} = \frac{x}{y}$
2. $xy\frac{dy}{dx} = y^2 + 9$
3. $\cos(y)\frac{dy}{dx} = \sin(x)$
4. $\frac{dy}{dx} = y^2 + 9$
5. $\frac{dy}{dx} = \frac{y^2+1}{x^2+1}$
6. $\frac{dy}{dx} = e^{2x-3y}$
7. $\frac{dy}{dx} = \frac{x}{y}$ *with* $y(1) = 3$
8. $\frac{dy}{dx} = 2x - 1 + 2xy - y$ *with* $y(0) = 2$
9. $y\frac{dy}{dx} = xy^2 + x$ *with* $y(0) = -2$
10. $y\frac{dy}{dx} = 3\sqrt{xy^2 + 9x}$ *with* $y(1) = 4$
11. $\frac{dy}{dx} = xy - 4x$
12. $\frac{dy}{dx} = xy - 3x - 2y + 6$
13. $\frac{dy}{dx} = \frac{y}{x}$
14. $(x^2+1)\frac{dy}{dx} = y^2 + 1$
15. $\frac{dy}{dx} = e^{-y}$
16. $\frac{dy}{dx} = 3xy^3$
17. $\frac{dy}{dx} - 3x^2y^2 = -3x^2$
18. $\frac{dy}{dx} = 200y - 2y^2$
19. $\frac{dy}{dx} = 3y^2 - y^2\sin(x)$
20. $\frac{dy}{dx} = \tan(y)$
21. $\frac{dy}{dx} = \frac{6x^2+4}{3y^2-4y}$
22. $(y^2-1)\frac{dy}{dx} = 4xy^2$
23. $\frac{dy}{dx} = e^{-y} + 1$
24. $\frac{dy}{dx} = \frac{2+\sqrt{x}}{2+\sqrt{y}}$
25. $\frac{dy}{dx} - 3x^2y^2 = 3x^2$
26. $\frac{dy}{dx} = \frac{y}{1+x}$ $y(0) = 1$
27. $\frac{dy}{dx} = -\frac{x}{y}$ $y(0) = 1$
28. $\frac{dy}{dx} = y^2 - 4$ $y(0) = -6$
29. $x\frac{dy}{dx} - y = 2x^2y$ $y(1) = e$
30. $\frac{dy}{dx} = \frac{x + xy^2}{4y}$ $y(1) = 0$

Solving First Order Linear Differential Equations Using an Integrating Factor

$$y' + p(x)\,y = q(x)$$

Wish to find $\boldsymbol{u}$ such that when I multiply the differential equation by $\boldsymbol{u}$ I can integrate both sides with respect to x. That is,

$\boldsymbol{u}\,y' + \boldsymbol{u}\,p\,y = \boldsymbol{u}\,q$. Choose $\boldsymbol{u}$ so that the left hand side is $(\boldsymbol{u}\,y)'$. This will be possible if $\boldsymbol{u}' = p\,\boldsymbol{u}$ or $\dfrac{d\boldsymbol{u}}{dx} = p(x)\boldsymbol{u}$. Use separation of variables to get $\boldsymbol{u} = e^{\int p(x)dx}$.

Example: solve $xy' - y = x^3$.

1. $y' - \dfrac{1}{x}y = x^2$
2. $\boldsymbol{u} = \dfrac{1}{x}$
3. $\dfrac{1}{x}y' - \dfrac{1}{x^2}y = x$ or $\left(\dfrac{1}{x}y\right)' = x$
4. $\displaystyle\int \left(\frac{1}{x}y\right)' = \int x\,dx \Rightarrow \frac{1}{x}y = \frac{1}{2}x^2 + C$

$$y = \frac{1}{2}x^3 + Cx$$

Solving First Order Linear Differential Equations Using an Integrating Factor

- **Find the general solution to each of the following first-order linear differential equations.**

a. $\frac{dy}{dx} + 2y = 6$

b. $\frac{dy}{dx} + 2y = 20e^{3x}$

c. $\frac{dy}{dx} = 4y + 16x$

d. $\frac{dy}{dx} - 2xy = x$

e. $x\frac{dy}{dx} + 3y - 10x^2 = 0$

f. $x^2\frac{dy}{dx} + 2xy = \sin(x)$

g. $x\frac{dy}{dx} = \sqrt{x} + 3y$

h. $\cos(x)\frac{dy}{dx} + \sin(x)\, y = \cos^2(x)$

i. $x\frac{dy}{dx} + (5x+2)y = \frac{20}{x}$

j. $2\sqrt{x}\frac{dy}{dx} + y = 2xe^{-\sqrt{x}}$

- **Find the solution to each of the following initial-value problems.**

a. $\frac{dy}{dx} - 3y = 6$ *with* $y(0) = 5$

b. $\frac{dy}{dx} - 3y = 6$ *with* $y(0) = -2$

c. $\frac{dy}{dx} + 5y = e^{-3x}$ *with* $y(0) = 0$

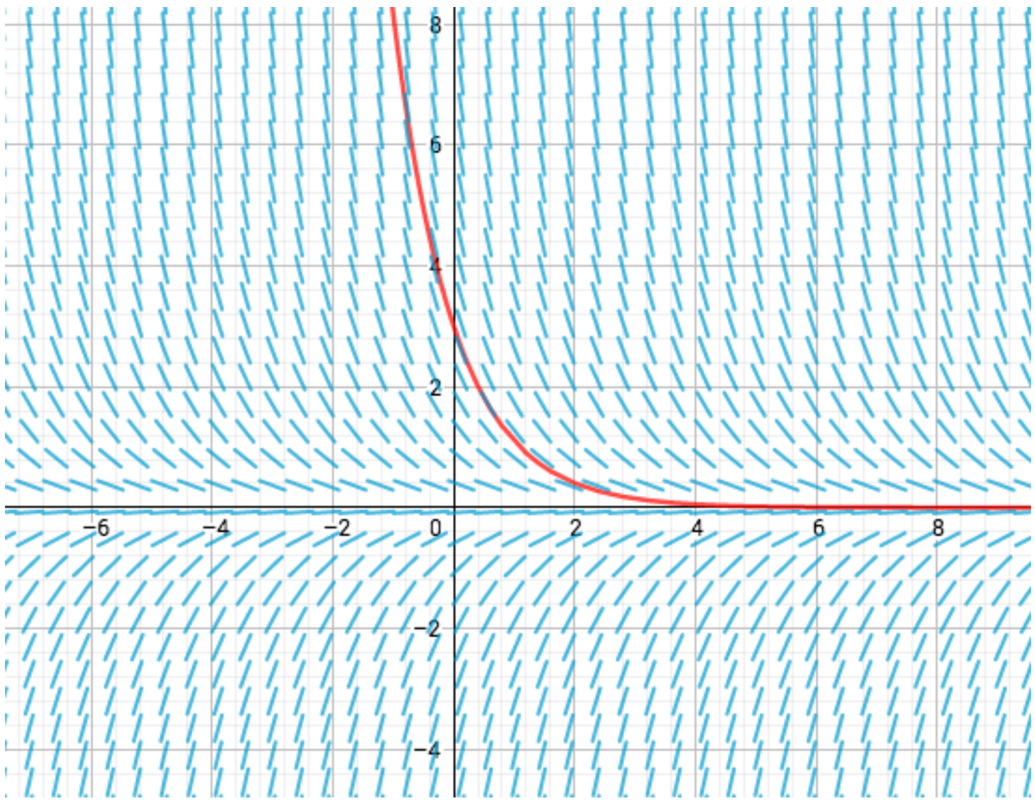
d. $2\frac{dy}{dx} + 3y = 20x^2$ *with* $y(1) = 10$

e. $x\frac{dy}{dx} = y + x^2\cos(x)$ *with* $y\left(\frac{\pi}{2}\right) = 0$

f. $(1+x^2)\frac{dy}{dx} = x\left[3 + 3x^2 - y\right]$ *with* $y(2) = 8$

Growth and Decay

$$\frac{dy}{dt} = ay$$



Growth and Decay

$$\frac{dy}{dt} = ay$$

Carbon 14 is a common form of carbon which decays over time. The amount of Carbon 14 contained in a preserved plant is modeled by the equation

$$\frac{dy}{dt} = -at \qquad \text{or} \qquad y(t) = y_0 e^{-at}$$

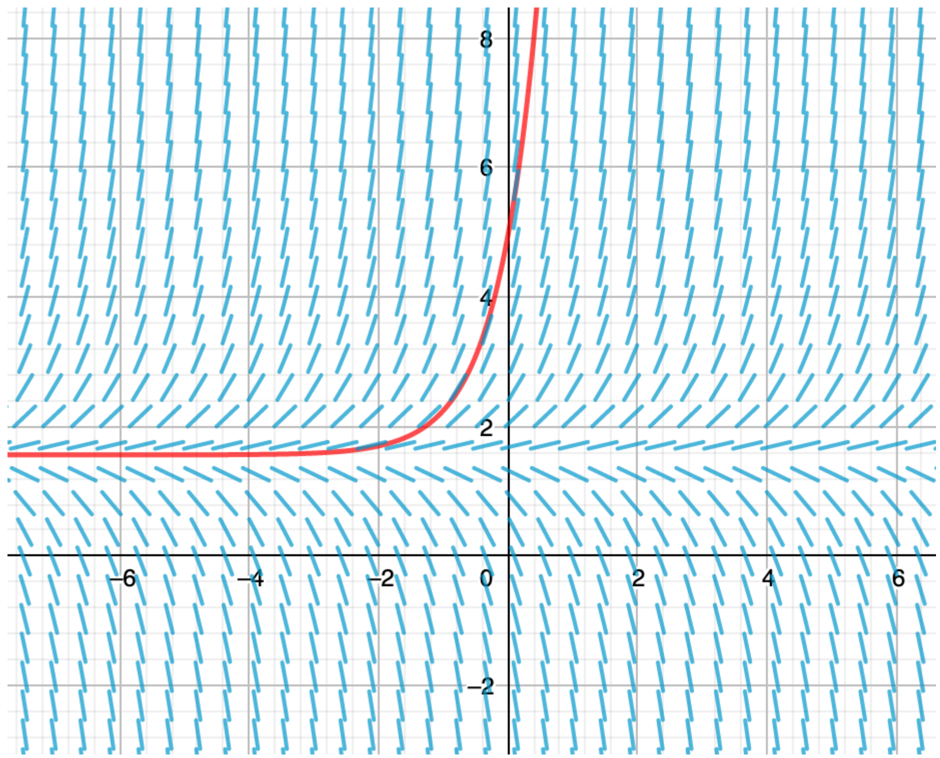
Time in this equation is measured in years from the moment when the plant dies ($t = 0$) and the amount of Carbon 14 remaining in the preserved plant is measured in micrograms (a microgram is one millionth of a gram). So when $t = 0$ the plant contains y_0 micrograms of Carbon 14.

a) The half-life of Carbon 14, that is the amount of time it takes for half of the Carbon 14 to decay, is approximately 5730 years. Use this information to find the constant a.

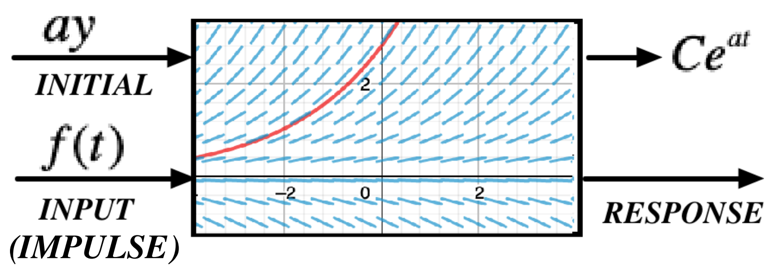
b) If there is currently 15% of the Carbon 14 remaining in the preserved plant, approximately when did the plant die ?

Growth and Decay

$$\frac{dy}{dt} = ay + b$$

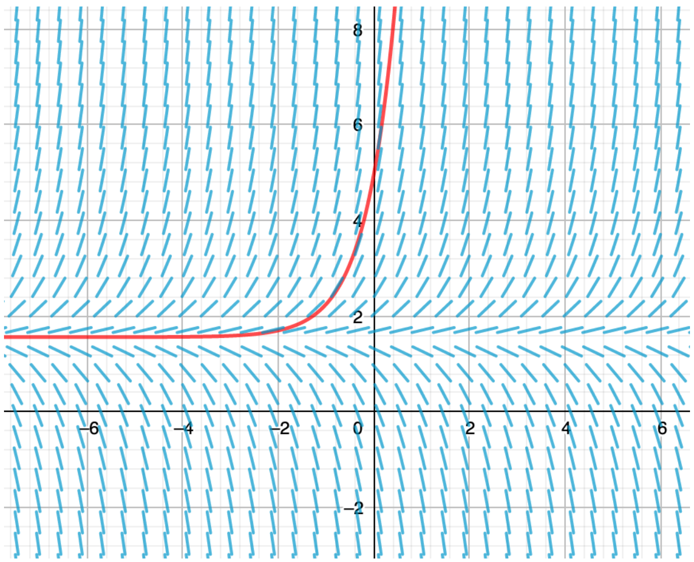


Growth and Decay
(Systems Model)



Growth and Decay

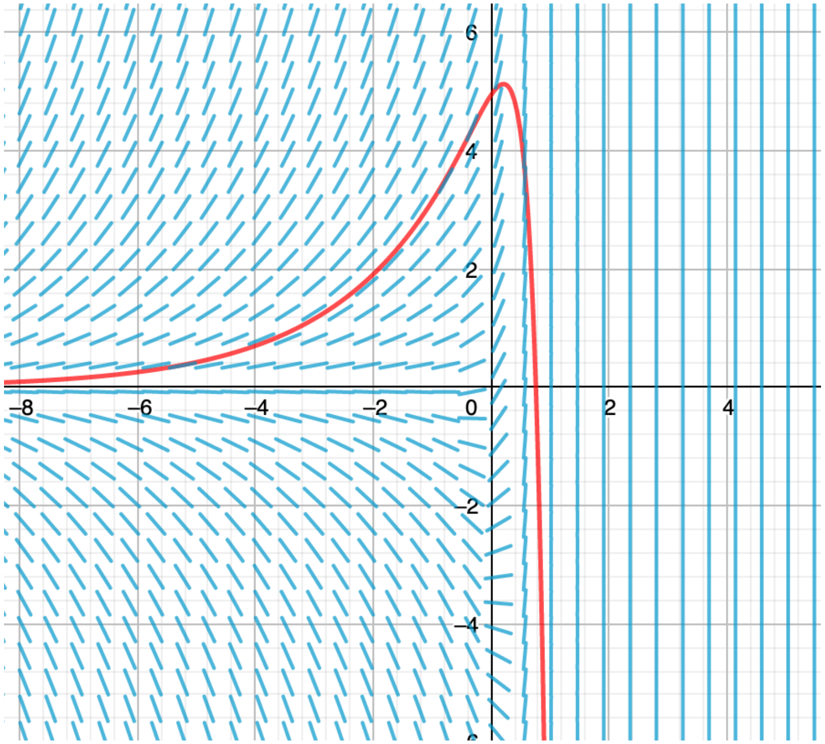
$$\frac{dy}{dt} = ay + b$$



$$y(t) = Ce^{at} - \frac{b}{a}$$

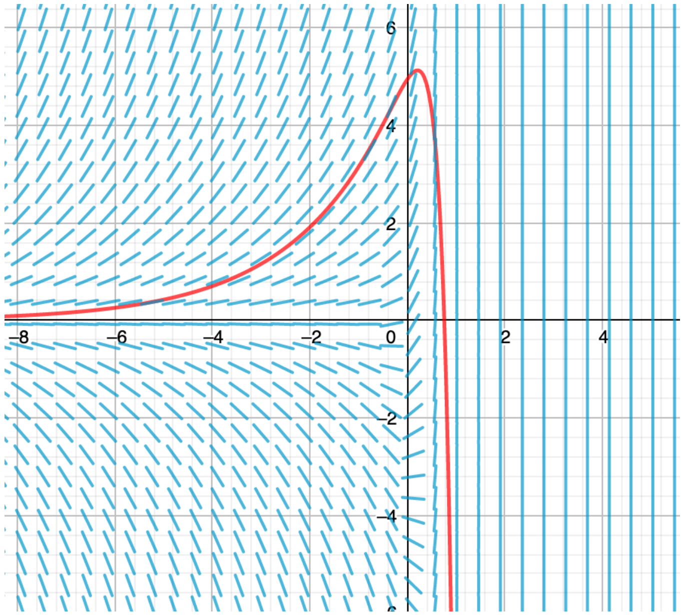
Growth and Decay

$$\frac{dy}{dt} = ay + e^{bt}$$



Growth and Decay

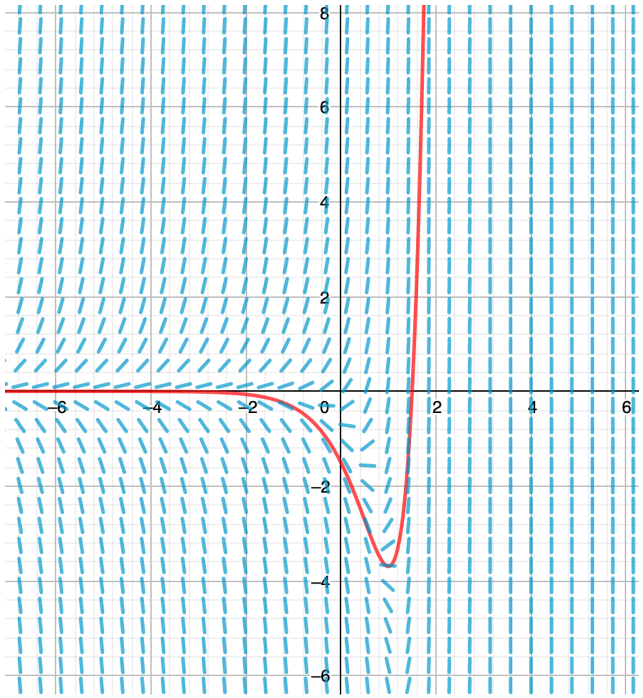
$$\frac{dy}{dt} = ay + e^{bt}$$



$$y(t) = Ce^{at} + \frac{1}{b-a}e^{bt}$$

Growth and Decay

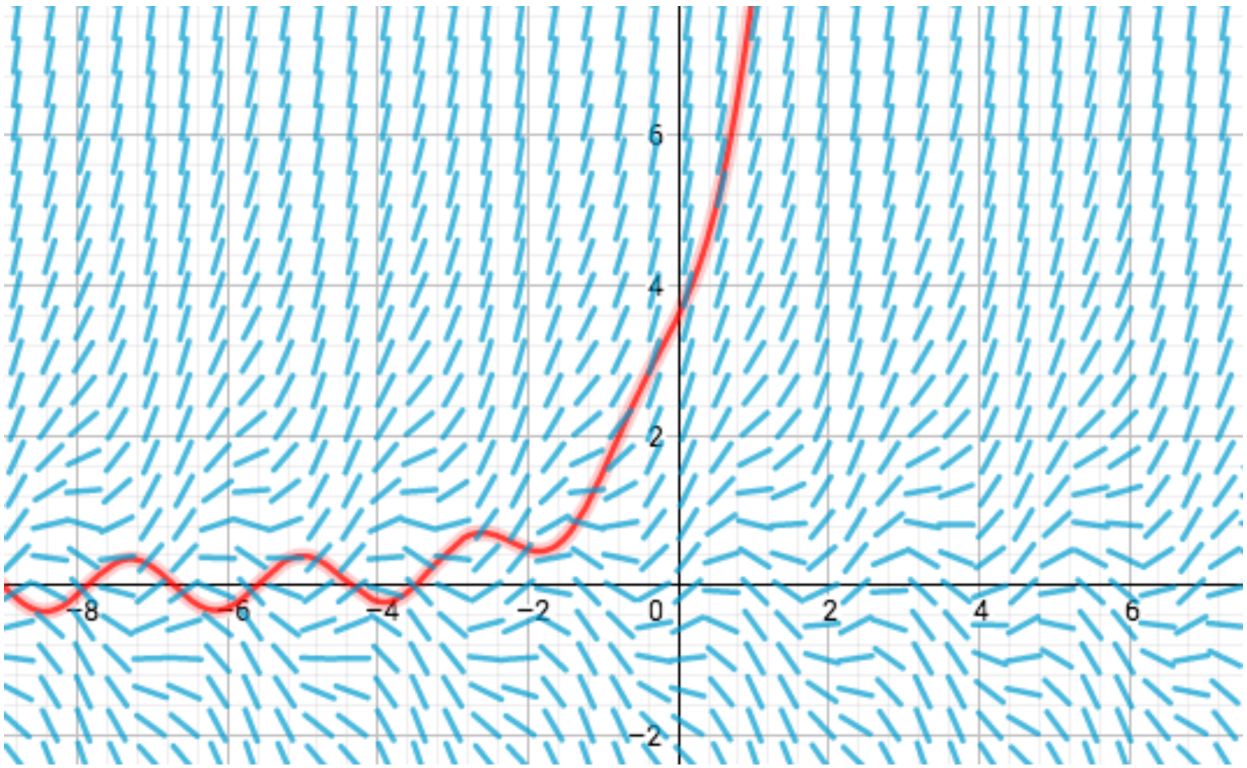
$$\frac{dy}{dt} = ay + e^{at}$$



$$y(t) = Ce^{at} + t\,e^{at}$$

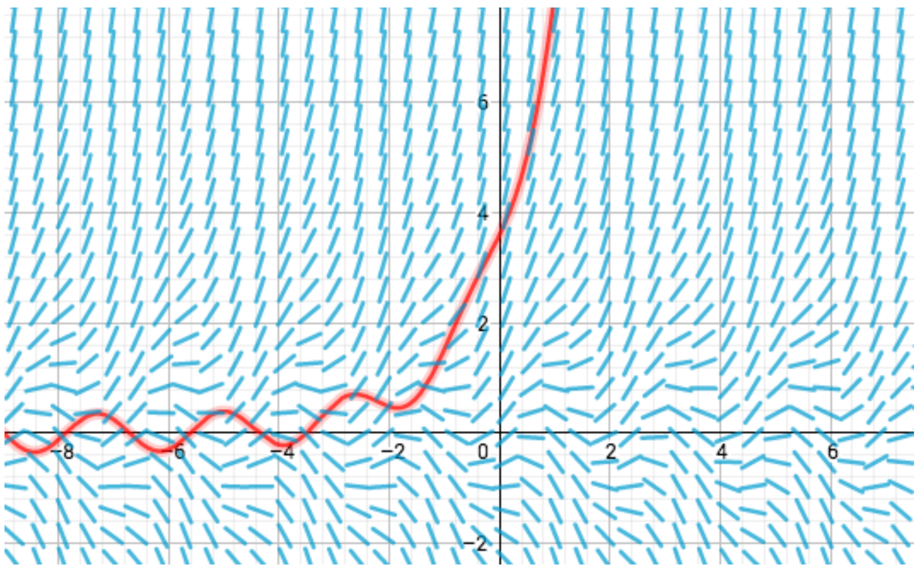
Growth and Decay

$$\frac{dy}{dt} = ay + \cos bt$$



Growth and Decay

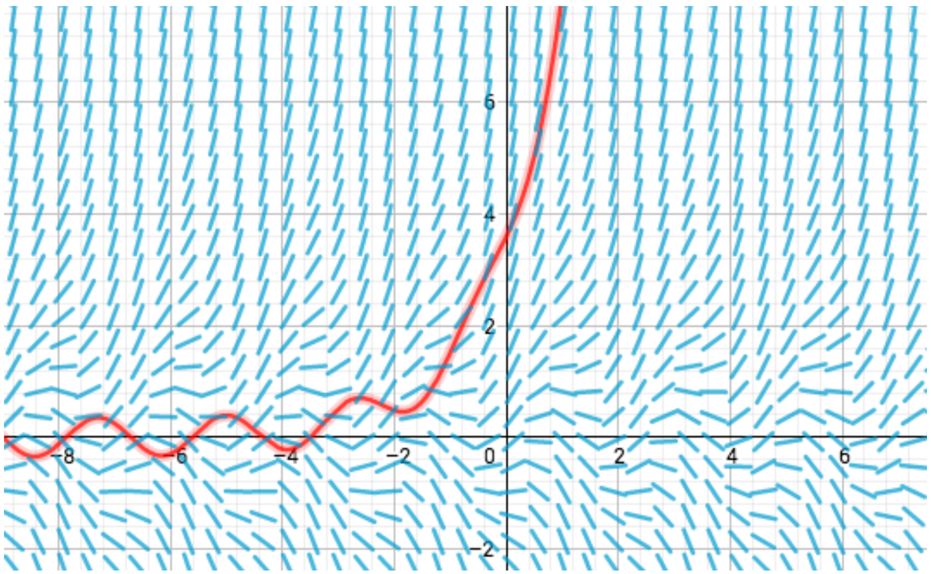
$$\frac{dy}{dt} = ay + \cos bt$$



$$y(t) = Ce^{at} + \frac{1}{a^2 + b^2}\left(b\sin bt - a\cos bt\right)$$

Growth and Decay

$$\frac{dy}{dt} = ay + \cos bt$$



$y_{null} = C\,e^{at}$ $\qquad y_p = M\cos bt + N\sin bt$

$$-bM\sin bt + bN\cos bt = a\left(M\cos bt + N\sin bt\right) + \cos bt$$
$$-bM\sin bt + bN\cos bt = aM\cos bt + aN\sin bt + \cos bt$$
$$-bM\sin bt + bN\cos bt \; - aM\cos bt - aN\sin bt = \cos bt$$

So,

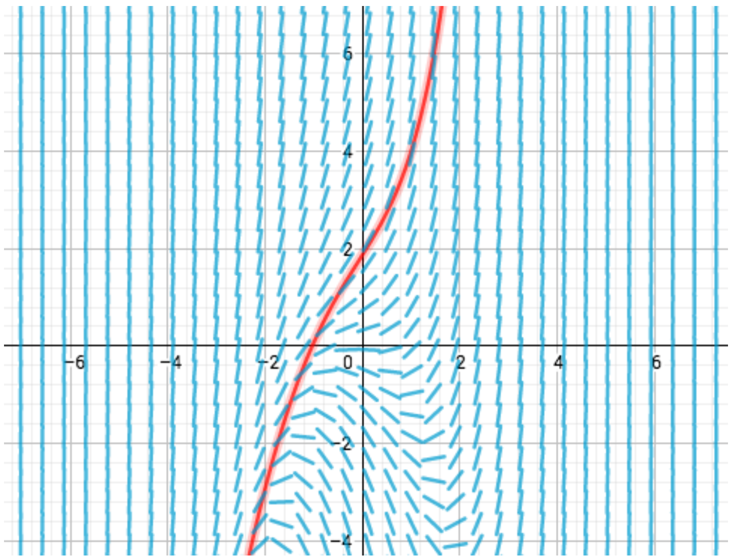
$-bM \; - \; aN = 0$

$-aM + bN = 1$

$$M = \frac{-a}{a^2+b^2},\quad N = \frac{b}{a^2+b^2}$$

$$\therefore\; y(t) = y_{null} + y_p = Ce^{at} \;+\; \frac{1}{a^2+b^2}\left(b\sin bt - a\cos bt\right)$$

Growth and Decay



$$\frac{dy}{dt} = ay + t^2$$

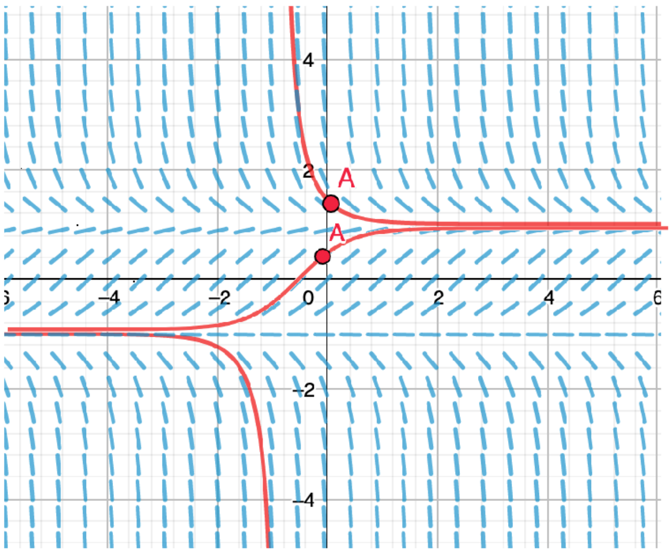
$$y = Ce^{at} + \left(-\frac{1}{a}t^2 - \frac{1}{a^2}t - \frac{1}{a^3} \right)$$

$$\frac{dy}{dt} = ay + t^n$$

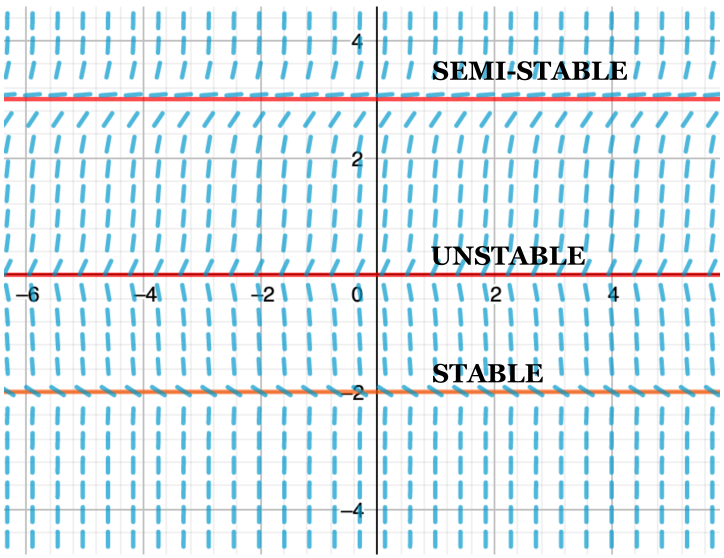
$$y = Ce^{at} + O\left(t^n\right)$$

Autonomous Differential Equations
(Equilibrium Solutions)

$$\frac{dy}{dx} = (1-y)(1+y)$$



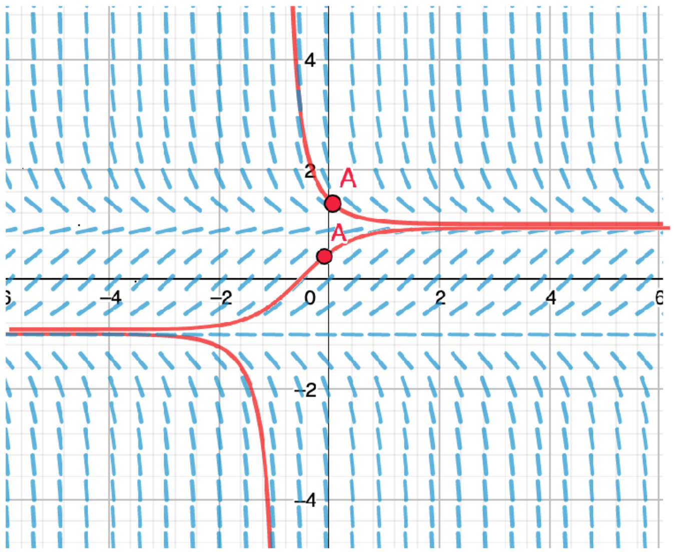
$$\frac{dy}{dx} = y(3-y)^2(2+y)$$



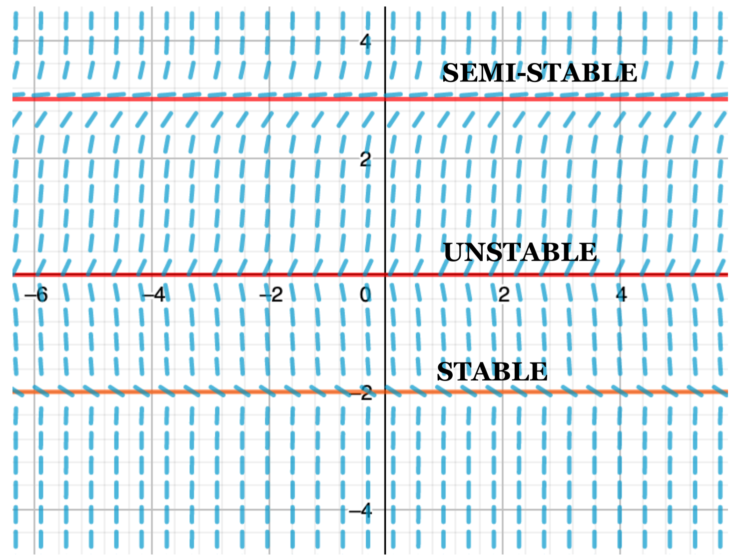
<u>Equillibrium solutions</u> are solutions to which other solutions tend to approach or in opposite moves away when $t \to \infty$. It's equilibrium because at those points solution doesn't depend on time (or any other variable you're integrating over). Those solutions that "attracts" other ones that started near them are called stable. Those that "push away" solutions that started near them are called unstable. And finally, in some cases there are solutions that either attracts or pushes other solutions depending on which side from them other solutions started are called semi-stable. The equilibrium solutions are values of y for which the differential equation says $dydt$=0. Therefore there are constant solutions at those values of y.

Autonomous Differential Equations
(Equilibrium Solutions)

$$\frac{dy}{dx} = (1-y)(1+y)$$

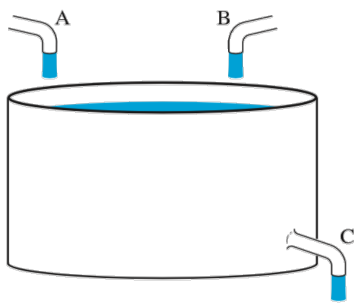


$$\frac{dy}{dx} = y(3-y)^2(2+y)$$



The stability of equillibrium solutions can also be found by evaluating the second derivative at the critical *y*-values. If *d/dy(dy/dx)* (y_c) < 0, then a stable solution exists.

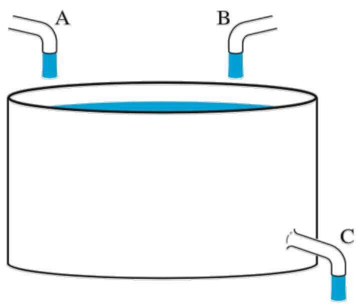
Mixing Models



Consider a large vat containing sugar water that is to be made into soft drinks. Suppose:

- The vat contains 100 gallons of liquid. Moreover, the amount flowing in is the same as the amount flowing out, so there are always 100 gallons in the vat.
- The vat is kept well mixed, so the sugar concentration is uniform throughout the vat.
- Sugar water containing 5 tablespoons of sugar per gallon enters the vat through pipe A at a rate of 2 gallons per minute.
- Sugar water containing 10 tablespoons of sugar per gallon enters the vat through pipe B at a rate of 1 gallon per minute.
- Sugar water leaves the vat through pipe C at a rate of 3 gallons per minute.

Mixing Models

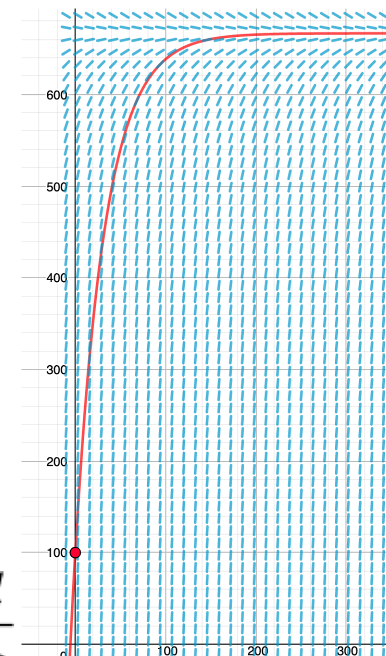


RATE-IN → $$\frac{dy}{dt} = RATE_{IN} - RATE_{OUT}$$ → RATE-OUT

$$RATE_{IN} - RATE_{OUT}$$

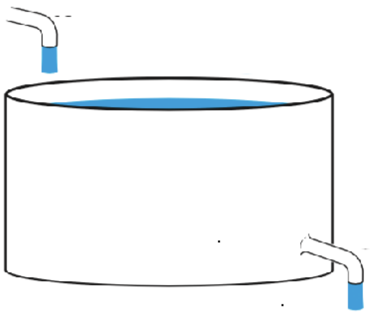
$$\text{concentration}_{IN} \cdot \text{flow\_rate}_{IN} - \text{concentration}_{OUT} \cdot \text{flow\_rate}_{OUT}$$

$$\frac{dy}{dt} = \left(5\frac{T}{gal} \cdot 2\frac{gal}{\min} + 10\frac{T}{gal} \cdot 1\frac{gal}{\min}\right) - \left(\frac{y}{100}\right)\frac{T}{gal} \cdot 3\frac{gal}{\min}$$



$$\frac{dy}{dt} = 20 - \frac{3}{100}y$$

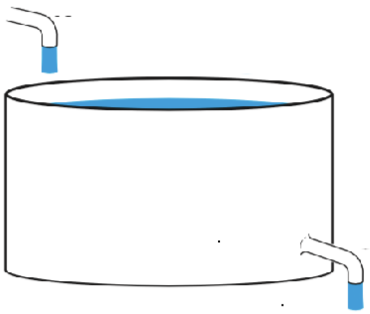
Mixing Models



A tank initially contains 50 gal of pure water. A salt solution containing 2 pounds of salt per gallon of water is poured into the tank at a rate of 3 gal/min. The mixture is stirred and is drained out of the tank at the same rate.

- Find the initial-value problem that describes the amount y of salt in the tank at any time.
- Find the amount of salt in the tank at any time.
- Find the amount of salt in the tank after 20 minutes.
- Find the amount of salt in the tank after a "long time."

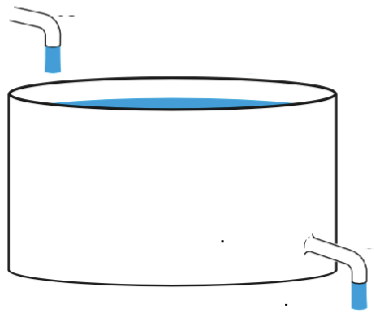
Mixing Models



A tank initially contains 100 gallons of water in which 10 pounds of salt are dissolved. A salt solution containing 0.5 pound of salt per gallon is poured into the tank at a rate of 1 gal/min. The mixture in the tank is stirred and drained off at the rate of 2 gal/min.

- Find the initial-value problem that describes the amount of salt $y(t)$ in the tank until the tank is empty.
- Find the amount of salt $y(t)$ in the tank until the tank is empty.
- Find the concentration $c(t)$ of salt in the tank until the tank is empty.
- Find the concentration of salt in the tank at the exact time the tank becomes empty.

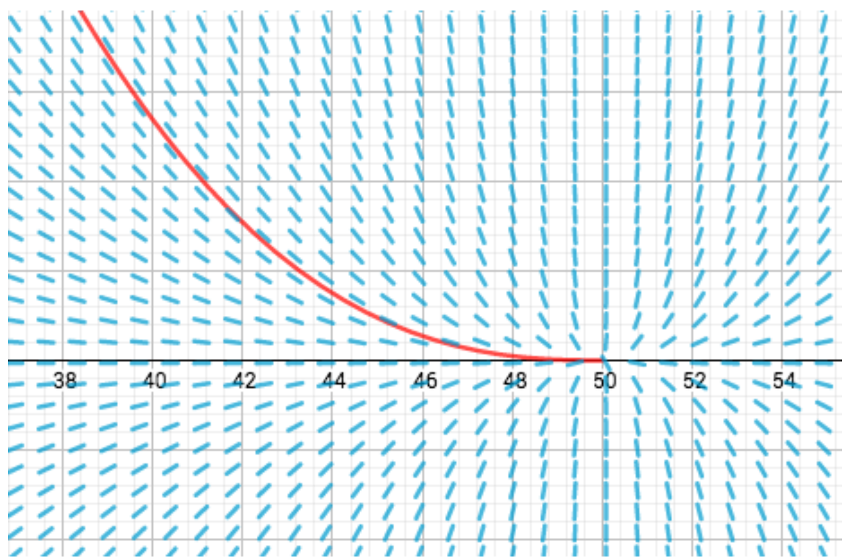
Mixing Models



A tank initially contains 100 gallons of brine whose salt concentration is 3 lb/gal. Fresh water is poured into the tank at a rate of 3 gal/min, and the well-stirred mixture flows out of the tank at the rate of 5 gal/min.

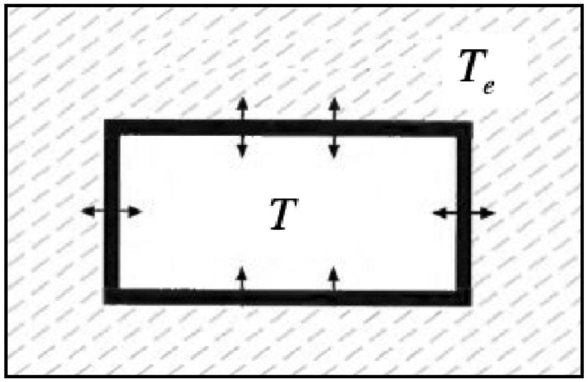
- Find the initial-value problem that describes the amount of salt in the tank.
- Find the amount of salt in the tank after time t .

$$\frac{dy}{dt} = \frac{-5y}{100-2t} \quad ; y(0) = 300$$



$$y(t) = \frac{3}{1000}(100-2t)^{\frac{5}{2}}$$

Conduction/Diffusion Models



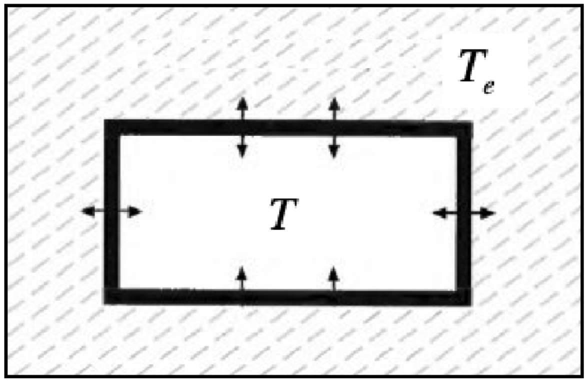
$$\frac{dT}{dt} \propto (T_e - T)$$

$$\frac{dT}{dt} = k(T_e - T)$$

At 12:00 midnight, with the temperature inside SLC at 70°F and the outside temperature at 20°F, the furnace breaks down. Two hours later the temperature in the building has fallen to 50°F.

- **Find the initial-value problem that describes the temperature inside the building for the remainder of the night. We assume that the outside temperature remains constant at 20°F.**
- **Determine the temperature in the building for the remainder of the night.**
- **Determine when the temperature in the building will fall to 40°F.**

Conduction/Diffusion Models



$$\frac{dT}{dt} \propto (T_e - T)$$

$$\frac{dT}{dt} = k(T_e - T)$$

$$\frac{dT}{dt} = k(20 - T);\ T(0) = 70^\circ F$$

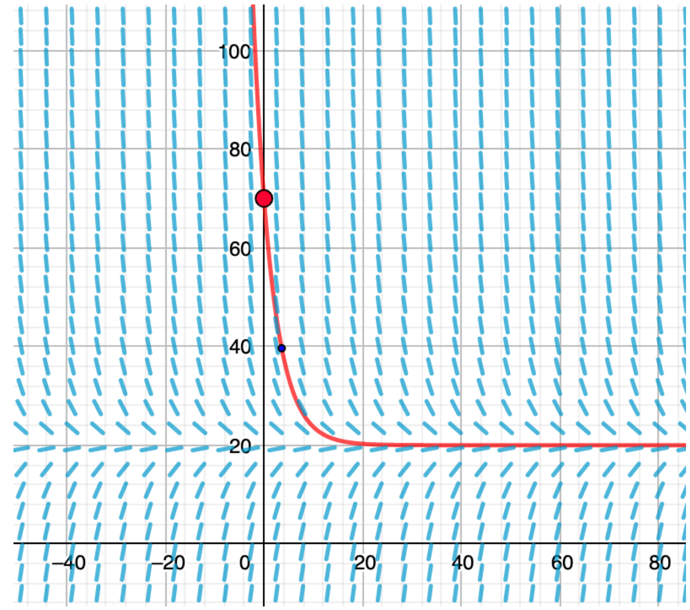
$$T(t) = 20 + 50e^{-kt}$$

To find k, we have $T(2) = 50$

$$T(t) = 20 + 50e^{-0.255t}$$

Solve for t,

$$40 = 20 + 50e^{-0.255t}, \quad t \approx 3.592 \text{ hr}$$



At 12:00 midnight, with the temperature inside SLC at 70°F and the outside temperature at 20°F, the furnace breaks down. Two hours later the temperature in the building has fallen to 50°F.

- **Find the initial-value problem that describes the temperature inside the building for the remainder of the night. We assume that the outside temperature remains constant at 20°F.**
- **Determine the temperature in the building for the remainder of the night.**
- **Determine when the temperature in the building will fall to 40°F.**

Kirchhoff's Law

The algebraic sum of all the voltage drops around an eletric loop or circuit is zero.

Voltage drops

$$RI = R\frac{dQ}{dt} \qquad L\frac{dI}{dt} \qquad \frac{Q}{C}$$

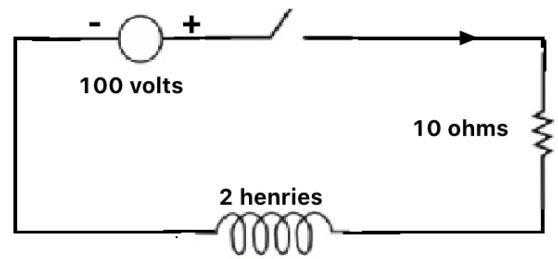
$$L\frac{dI}{dt} + RI = E \qquad\qquad R\frac{dQ}{dt} + \frac{Q}{C} = E$$

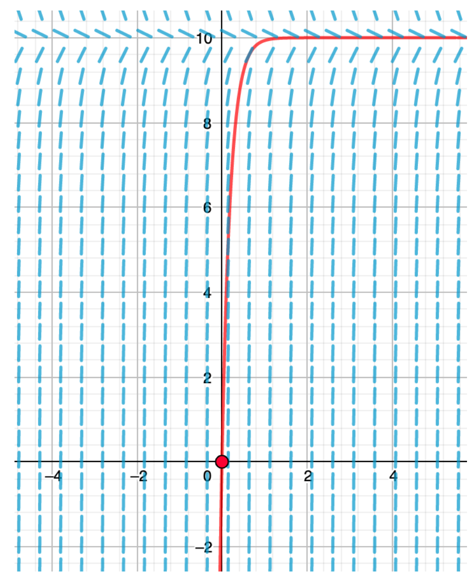
Kirchhoff's Law

The algebraic sum of all the voltage drops around an eletric loop or circuit is zero.

- **A generator having emf 100 volts is connected in series with a 10 ohm resistor and an inductor of 2 henries. If the switch is closed at time $t = 0$, set up a differential equation for the current and determine the current at time t.**

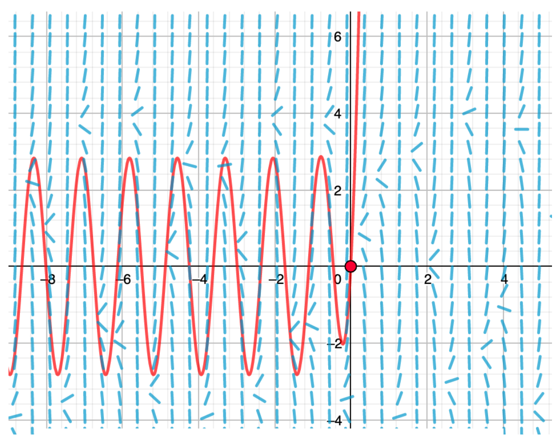
We have the voltage supplied at 100 volts, the voltage drop across the resistance (RI) = 10 volts, and the voltage drop across the inductor ($L\, dI/dt$) = 2 dI/dt.





$$I(t) = 10 + C e^{-5t}$$

- **Replace the 100 volt generator in the first example by one having an emf of 20 cos(5t) volts.**

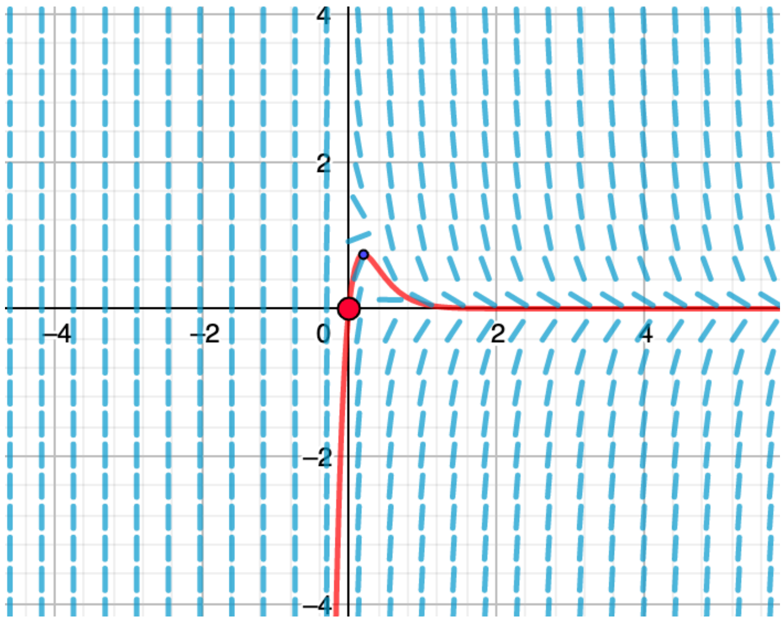


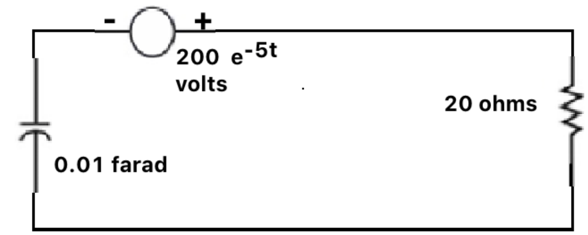
$$I(t) = C e^{-5t} + \cos 5t + \sin 5t$$

Kirchhoff's Law

The algebraic sum of all the voltage drops around an eletric loop or circuit is zero.

- **A decaying emf $E = 200e^{-5t}$ is connected in series with a 20 ohm resistor and a 0.01 farad capacitor. Assuming $Q = 0$ at $t = 0$, find the charge and the current at any time t. Show that the charge reaches a maximum, calculate it and find when it is reached.**





$$R\frac{dQ}{dt} + \frac{Q}{C} = E$$

$$20\,Q' + \frac{Q}{0.01} = 200\,e^{-5t}$$

$$Q' + 5Q = 10\,e^{-5t}$$

$$Q(t) = 10t\,e^{-5t} + Ce^{-5t}$$

$$\frac{dQ}{dt} = 0 \text{ when } t = \frac{1}{5} \quad \text{and} \quad Q\left(\frac{1}{5}\right) = \frac{2}{e} \approx 0$$