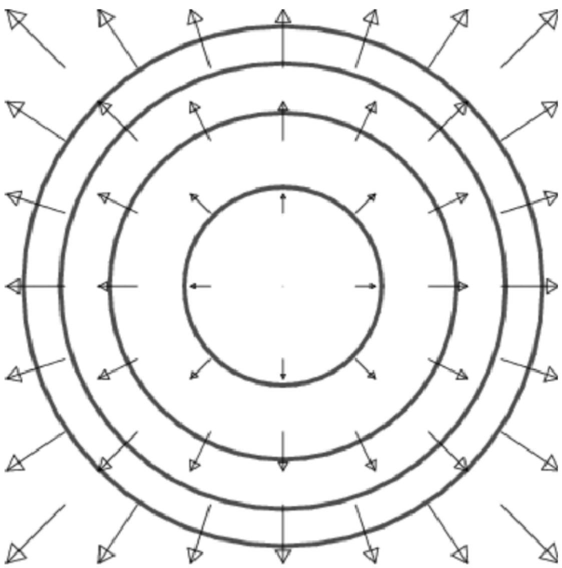
Exact Equations



An **exact differential equation** is one that represents the total differential of a scalar function (a potential function), meaning its solution curves are level sets of that potential function. This occurs when the associated vector field is conservative, a field that is the gradient of a potential function, ensuring path-independent line integrals. The relationship is tested by checking if the partial derivative of the first component of the vector field with respect to the second variable equals the partial derivative of the second component with respect to the first.

Conservative Fields and Potential Functions

- Conservative Field: A vector field F is conservative if it is the gradient of a scalar potential function, $\Phi(x, y)$, such that $F = \nabla\Phi$.

- Relationship to Exact Differential Equations: If a vector field $F = \langle M, N\rangle$ is conservative, then the differential form Mdx + Ndy is exact.

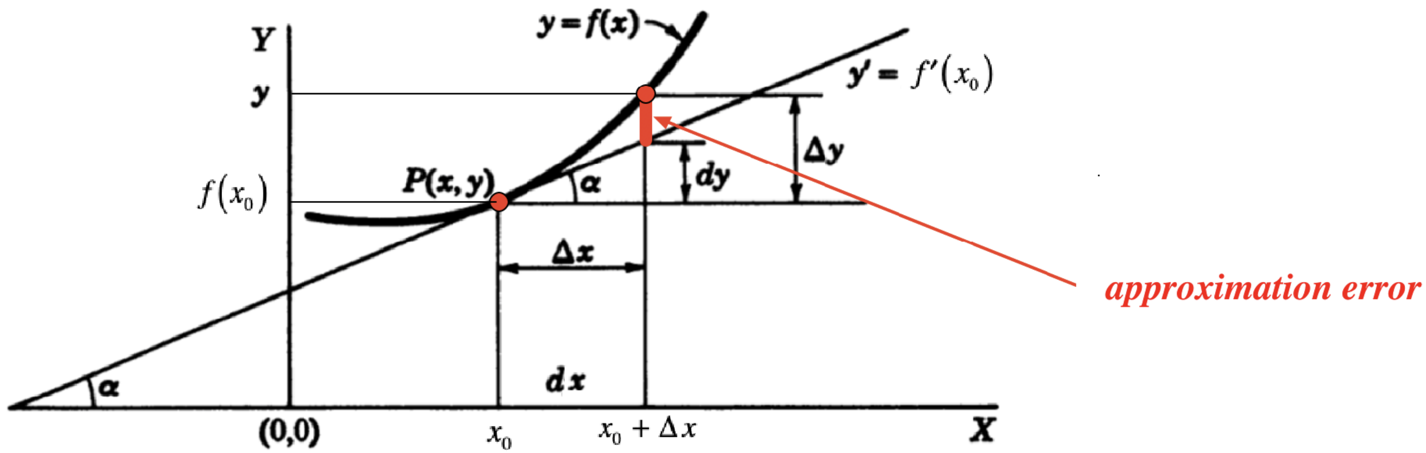
- Potential Function: The scalar function Φ is the potential of the conservative field.

- Path Independence: A key property of conservative fields is that the line integral of the field between two points is independent of the path taken. The work done by the field depends only on the endpoints.

In summary, an exact differential equation signifies a conservative field, and the process of finding the solution to the equation is equivalent to finding the potential function of that conservative field.

Exact Equations

$$y - f(x_0) = f'(x_0)(x - x_0)$$

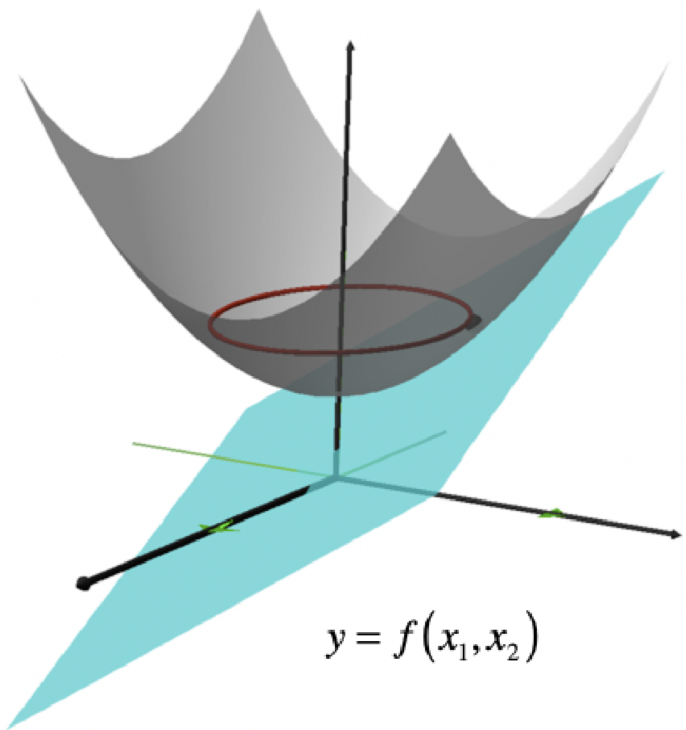


$$\frac{dy}{dx} = \frac{y - f(x_0)}{x - x_0} = f'(x_0) \text{ or}$$

$$y - f(x_0) = f'(x_0)(x - x_0) = f'(x_0)\,dx$$

$$dy = f'(x_0)\,dx$$

Exact Equations



$$y = f(x_1, x_2)$$

$$y - f(x_{1_0}, x_{2_0}) = f_1'(x_{1_0}, x_{2_0})(x_1 - x_{1_0}) + f_2'(x_{1_0}, x_{2_0})(x_2 - x_{2_0})$$

$$dy = f_1'(x_{1_0}, x_{2_0})dx_1 + f_2'(x_{1_0}, x_{2_0})dx_2$$

Exact Equations

Example: $z = f(x, y) = 3x^2y + 5xy + y^3 + 5$ then the differential of z is

$$dz = \frac{\partial f(x,y)}{\partial x}dx + \frac{\partial f(x,y)}{\partial y}dy = (6xy + 5y)dx + (3x^2 + 5x + 3y^2)dy$$

If therefore we started with the differential expression

$(6xy + 5y)dx + (3x^2 + 5x + 3y^2)dy$ we would know that it is the total differential of $f(x, y)$,

A differential expression $M(x,y)dx + N(x,y)dy$ is called an **exact differential** if it is the total differential of a function $f(x, y)$, that is

$$M(x,y) = \frac{\partial}{\partial x}f(x,y) \quad \text{and} \quad N(x,y) = \frac{\partial}{\partial y}f(x,y)$$

Setting $(6xy + 5y)dx + (3x^2 + 5x + 3y^2)dy$ equal to zero, we obtain the differential equation

$$(6xy + 5y)dx + (3x^2 + 5x + 3y^2)dy = 0$$

whose solution is

$$f(x, y) = 3x^2y + 5xy + y^3 = C$$

Exact Equations

$$M\,dx + N\,dy = dU \quad \text{with} \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$M = \frac{\partial U}{\partial x} \quad \text{and} \quad N = \frac{\partial U}{\partial y}$$

To find U:

If $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ then

$U = \displaystyle\int M\,\partial x + f(y)$ then substitute U in $\dfrac{\partial U}{\partial y} = N$ to resolve $f(y)$

Solving exact ODEs (algorithm)

$Mdx + Ndy = 0$ when $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

1. $U = \int M\,\partial x$
2. **Find** $\dfrac{\partial U}{\partial y}$
3. **Set** $\dfrac{\partial U}{\partial y} = N$ **to resolve** $f(y)$
4. **Set** $U = C$

Exact Equations

$$M\,dx + N\,dy = dU \quad \text{with} \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Example:

$$2xy\,dx + \left(x^2 + \cos y\right)dy = 0$$

$$M = 2xy \qquad N = x^2 + \cos y$$

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$$

Therefore U exists such that

$$\frac{\partial U}{\partial x} = 2xy\ , \quad \frac{\partial U}{\partial y} = x^2 + \cos y$$

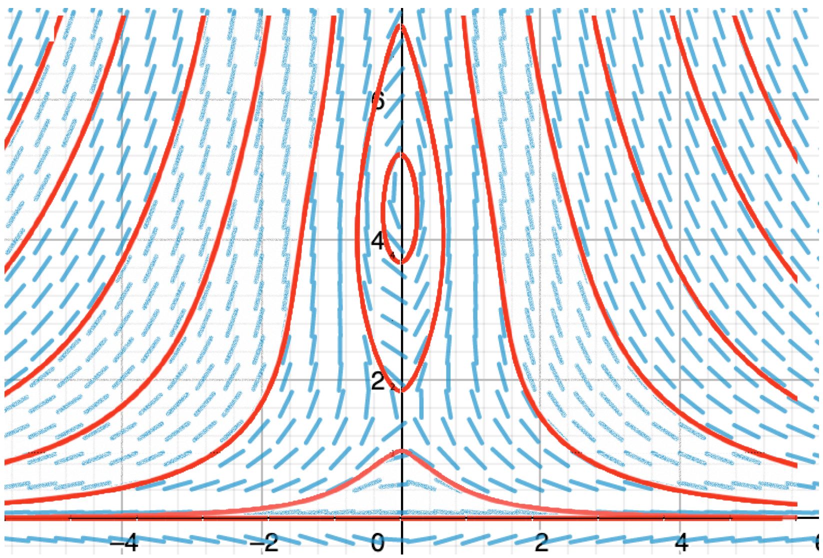
Integrating the first equation with respect to x gives $U = x^2 y + f(y)$

Now, substituting into the second equation, we have

$$x^2 + f'(y) = x^2 + \cos y$$

$$\Rightarrow f'(y) = \cos y \quad \text{or} \quad f(y) = \sin y$$

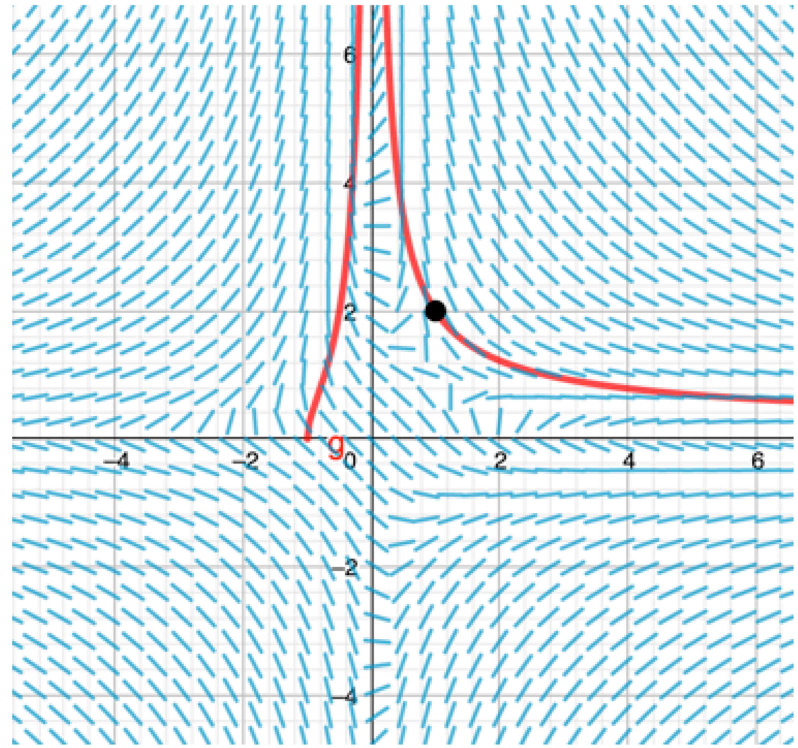
Hence $U = x^2 y + \sin y$ and the solution is $x^2 y + \sin y = C$



Exact Equations

$$M\,dx + N\,dy = dU \quad \text{with} \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

- Solve: $y' = \dfrac{xy^2 - 1}{1 - x^2 y}$ given $y(1) = 2$



$$(xy^2 - 1)dx + (x^2y - 1)dy = 0,$$

$$M = xy^2 - 1 \qquad N = x^2y - 1$$

$$\frac{\partial}{\partial y}(xy^2 - 1) = 2xy = \frac{\partial}{\partial x}(x^2y - 1)$$

$$\int (xy^2 - 1)\partial x = \frac{1}{2}x^2y^2 - x + f(y)$$

$$\frac{\partial}{\partial y}\left(\frac{1}{2}x^2y^2 - x + f(y)\right) = x^2y + f'(y)$$

Now $x^2y + f'(y) = x^2y - 1$

$$\therefore \int f'(y)dy = f(y) = -y$$

So $U = \frac{1}{2}x^2y^2 - x - y$ and $\frac{1}{2}x^2y^2 - x - y = C$

$y(1) = 2 \Rightarrow C = -1$

Solving Exact Differential Equations

- **Find the general solution to each of the following exact differential equations.**

a. $2xy + y^2 + [2xy + x^2]\dfrac{dy}{dx} = 0$

b. $2xy^3 + 4x^3 + 3x^2y^2\dfrac{dy}{dx} = 0$

c. $2 - 2x + 3y^2\dfrac{dy}{dx} = 0$

d. $1 + 3x^2y^2 + [2x^3y + 6y]\dfrac{dy}{dx} = 0$

e. $4x^3y + [x^4 - y^4]\dfrac{dy}{dx} = 0$

f. $1 + \ln|xy| + \dfrac{x}{y}\dfrac{dy}{dx} = 0$

g. $1 + e^y + xe^y\dfrac{dy}{dx} = 0$

h. $e^y + [xe^y + 1]\dfrac{dy}{dx} = 0$

Solving Exact Differential Equations

f.

$$(1+\ln xy)dx+\left(\frac{x}{y}\right)dy=0,$$

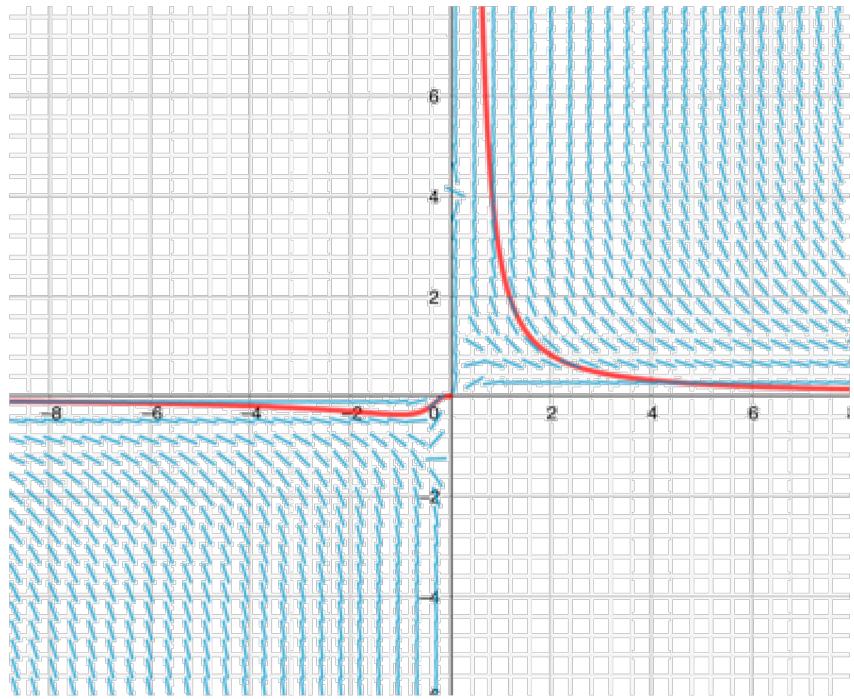
$$M=1+\ln xy \qquad N=\frac{x}{y}$$

$$\frac{\partial}{\partial y}(1+\ln xy)=\frac{1}{y}=\frac{\partial}{\partial x}\left(\frac{x}{y}\right)$$

$$\int(1+\ln xy)\partial x=x\ln xy+f(y)$$

$$\frac{\partial}{\partial y}(x\ln xy+f(y))=\frac{x}{y}+f'(y)$$

So the $\frac{x}{y}+f'(y)=\frac{x}{y}$



$$\therefore f'(y)\Rightarrow f(y)=0 \quad \text{and} \quad x\ln xy=C$$