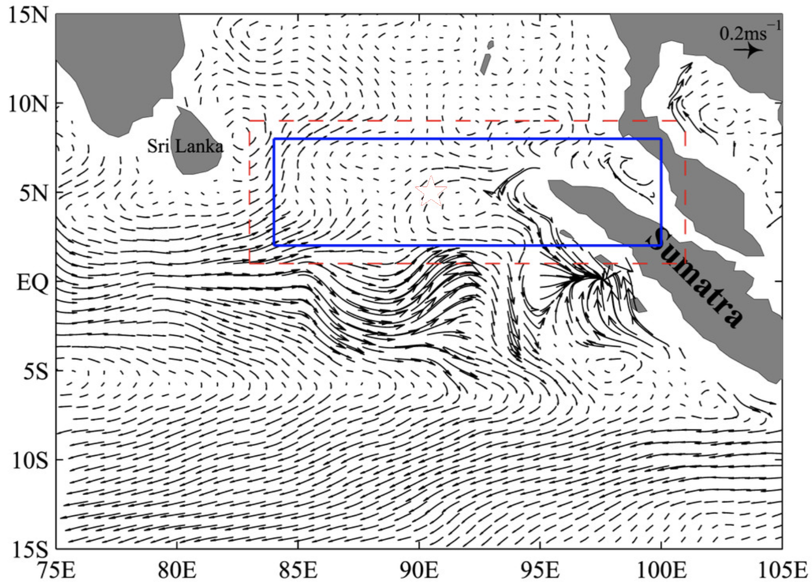
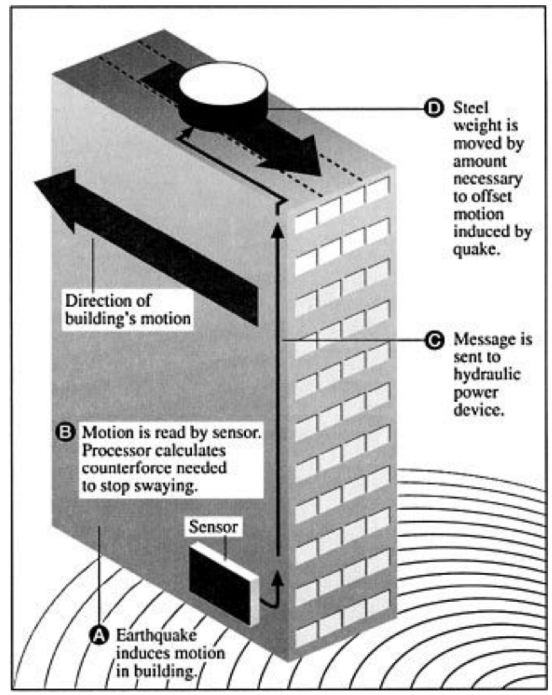
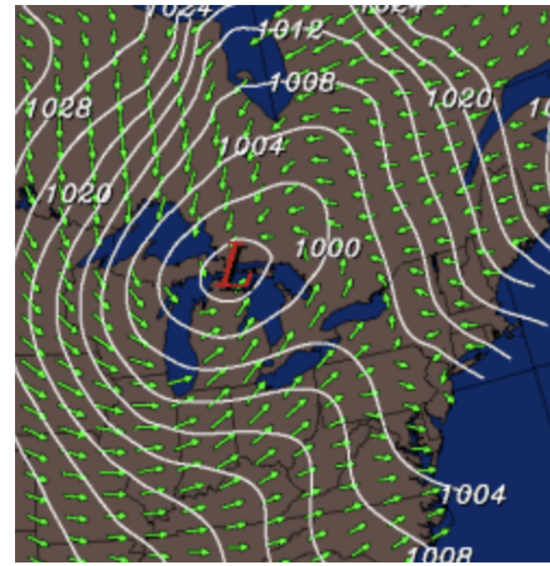
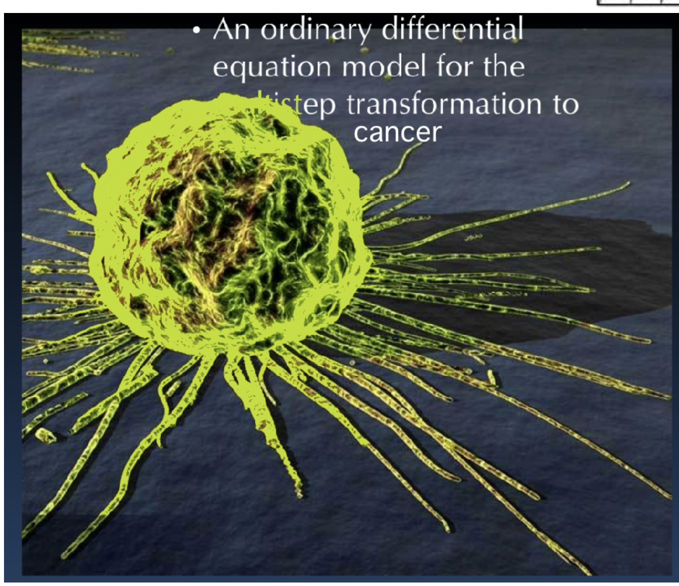
MTH 211 Ordinary Differential Equations









$$\frac{dy}{dt} = \left(\mathrm{diag}\left(\mathrm{diag}\left(y^T k\right)^T b\right) M + \mathrm{diag}\left((b-d)^T y\right)\right) S \left(1 - a(y)\frac{P_{NM}}{10^6}\right) \times \left(1 - \frac{P_{NM}}{10^{13}}\right) + m_m,$$

$$\frac{dP_N}{dt} = 0,$$

$$\frac{dP_A}{dt} = \left(\frac{100 P_N k_1}{b} - \frac{(3\times 100 + 3\times 10 + 1) P_A k_1}{b}\right) \times \left(1 - \frac{P_{NM}}{10^6}\right)\left(1 - \frac{P_{NM}}{10^{13}}\right)$$

$$\frac{dP_D}{dt} = \left(\frac{100 P_N k_1}{b} + P_D\left(\frac{1}{b} - \frac{1}{d_D}\right) - \frac{(3\times 100 + 3\times 10 + 1) P_D k_1}{b}\right) \times \left(1 - \frac{P_{NM}}{10^6}\right)\left(1 - \frac{P_{NM}}{10^{13}}\right)$$

$$\frac{dP_G}{dt} = \left(\frac{100 P_N k_1}{b} - \frac{(3\times 100 + 3\times 10 + 1) P_G k_2}{b}\right) \times \left(1 - \frac{P_{NM}}{10^6}\right)\left(1 - \frac{P_{NM}}{10^{13}}\right)$$

$$\frac{dP_R}{dt} = \left(\frac{100 P_N k_1}{b} + P_R\left(\frac{1}{b_R} - \frac{1}{d}\right) - \frac{(3\times 100 + 3\times 10 + 1) P_R k_1}{b_R}\right) \times \left(1 - \frac{P_{NM}}{10^6}\right)\left(1 - \frac{P_{NM}}{10^{13}}\right)$$

mth_211

- A rock falls from the top of a cliff and hits the ground at the base of the cliff at a speed of 160 ft/s. How high is the cliff?

$$\frac{dv}{dt} = a(t) = -g = -32 \text{ ft/s}^2 \quad \text{so} \quad v(t) = \int \frac{dv}{dt} = \int -32\, dt = 32t + C$$

$$v(0) = 0, \text{ so } v(t) = 32t \quad \text{and} \quad v(t) = 160 \text{ when } t = 5$$

$$s(t) = \int \frac{ds}{dt} = \int v\, dt = \int 32t\ dt = 16t^2 + C$$

$$s(0) = 0, \text{ so } s(t) = 16t^2 \quad \text{and} \quad s(5) = 400 \text{ ft}$$

- With full brakes applied, a freight train can decelerate at a constant rate of $\frac{1}{6}$ m/s^2.

How far will the rain travel while braking to a full stop from an initial speed of 60 km/hr?

First of all $-\frac{1}{6}\text{ m/s}^2 = -\frac{1}{6000}\text{ km/s}^2$ and $60\text{ km/h} = \frac{1}{60}\text{ km/s}$

$$\frac{dv}{dt} = a(t) = -\frac{1}{6000}\text{km/s} \quad \text{so} \quad v(t) = \int \frac{dv}{dt} = \int -\frac{1}{6000}dt = -\frac{1}{6000}t + C$$

$$v(0) = \frac{1}{60}, \text{ so } v(t) = -\frac{1}{6000}t + \frac{1}{60}, \text{ and } v(t) = 0 \text{ when } t = 100$$

$$s(t) = \int \frac{ds}{dt} = \int v\,dt = \int -\frac{1}{6000}t + \frac{1}{60}\,dt = -\frac{1}{12000}t^2 + \frac{1}{60}t + C$$

$$s(0) = 0, \text{ so } s(t) = -\frac{1}{12000}t^2 + \frac{1}{60}t \quad \text{and} \quad s(100) = \frac{5}{6}\text{km}$$

161. Important General Statement. — Strictly speaking, there is no such thing as a Process of Integration. Whenever a differential is proposed for integration, the first question is, *Is this a Known Form*? That is. *Can we see by inspection what function, being differentiated, produces this?* If we cannot thus discern the integral by a simple inspection, the only question remaining is, *Can we transform the differential into an equivalent expression the integral of which we can recognize?* Thus, in any case, we pass from the differential to its integral by a simple inspection; and the sufficient reason always is, *This expression is the integral of that, because, being differentiated, it produces it.*

... from Olney's *General Geometry and Calculus* (1871)

- *Find a general solution for each of the following directly integrable equations.*

a. $\frac{dy}{dx} = 4x^3$

b. $\frac{dy}{dx} = 20e^{-4x}$

c. $x\frac{dy}{dx} + \sqrt{x} = 2$

d. $\sqrt{x+4}\,\frac{dy}{dx} = 1$

e. $\frac{dy}{dx} = x\cos\left(x^2\right)$

f. $\frac{dy}{dx} = x\cos(x)$

g. $x = \left(x^2 - 9\right)\frac{dy}{dx}$

h. $1 = \left(x^2 - 9\right)\frac{dy}{dx}$

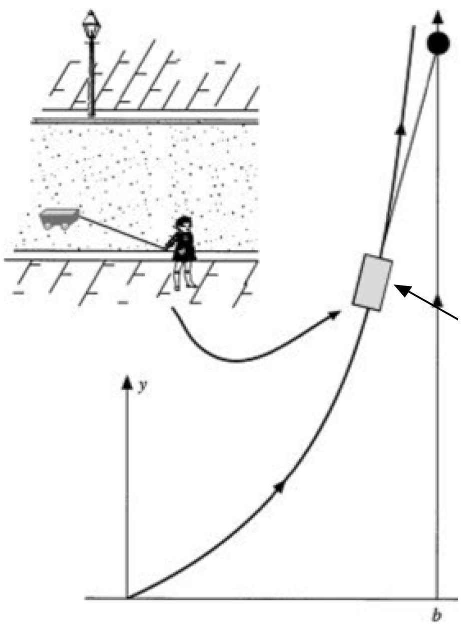
i. $1 = x^2 - 9\frac{dy}{dx}$

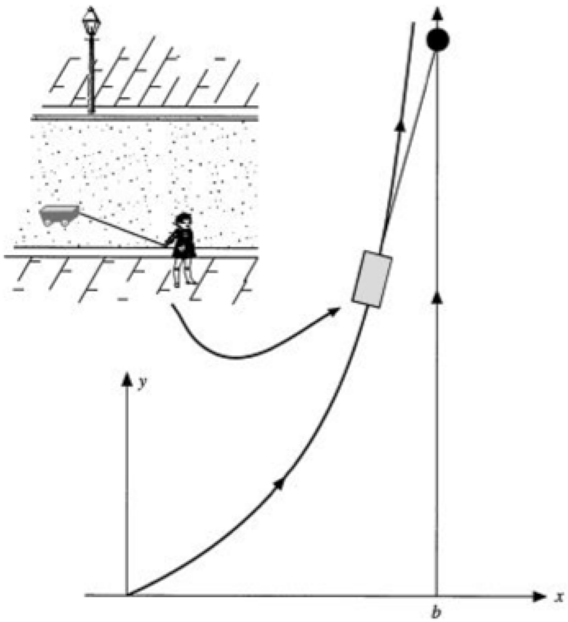
j. $\frac{d^2y}{dx^2} = \sin(2x)$

k. $\frac{d^2y}{dx^2} - 3 = x$

l. $\frac{d^4y}{dx^4} = 1$

If a child walks in a straight line and pulls a wagon attached at the end of a rope, the wagon traces out a curve called the **tractrix**. It is the same curve that is traced out by an airplane as it chases another airplane moving in a straight line. Tractrices also occur in many other areas of applied mathematics.



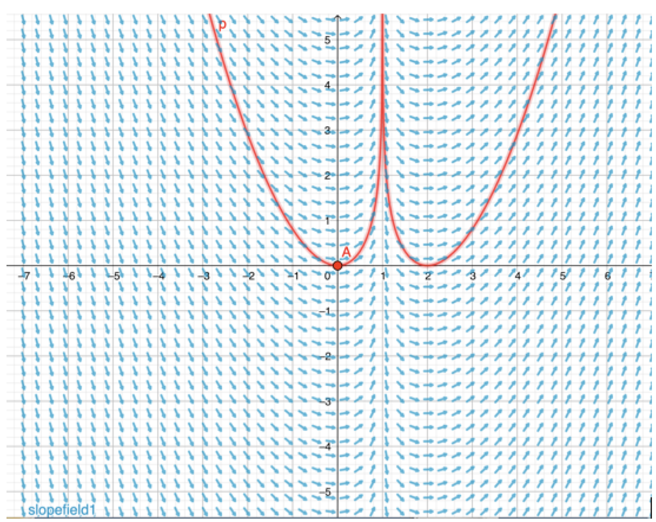


$$y-(x-b)\frac{dy}{dx}=\int_0^x \sqrt{1+\left[y'(s)\right]^2}\,ds$$

$$(x-b)\frac{dw}{dx}=-\sqrt{1+w^2} \qquad \text{where} \quad w=\frac{dy}{dx}.$$

$$w(x)=\frac{dy}{dx}=\frac{1}{2}\left(\left(1-\frac{x}{b}\right)^{-1}-\left(1-\frac{x}{b}\right)\right) \text{ and solving for } y(x)$$

$$y(x)=\frac{b}{2}\left[\frac{1}{2}\left(\left(1-\frac{x}{b}\right)^{2}-1\right)-\ln\left(1-\frac{x}{b}\right)\right]$$



- *Solve each of the following initial-value problems*

a. $\dfrac{dy}{dx} = 4x + 10e^{2x}$ *with* $y(0) = 4$

b. $\sqrt[3]{x+6}\,\dfrac{dy}{dx} = 1$ *with* $y(2) = 10$

c. $\dfrac{dy}{dx} = \dfrac{x-1}{x+1}$ *with* $y(0) = 8$

d. $x\dfrac{dy}{dx} + 2 = \sqrt{x}$ *with* $y(1) = 6$

e. $\cos(x)\dfrac{dy}{dx} - \sin(x) = 0$ *with* $y(0) = 3$

f. $\left(x^2+1\right)\dfrac{dy}{dx} = 1$ *with* $y(0) = 3$

g. $x\dfrac{d^2y}{dx^2} + 2 = \sqrt{x}$ *with* $y(1) = 8$ *and* $y'(1) = 6$

$$F = F_{\text{grav}} + F_{\text{air}}$$

$$F_{\text{air}}(v) = -\gamma v$$

where γ is some positive value. The actual value of γ will depend on such parameters a object's size, shape, and orientation, as well as the density of the air through which the obje moving.

$$F = F_{\text{grav}} + F_{\text{air}} = -9.8m + \gamma v$$

$$F = m\frac{dv}{dt}$$

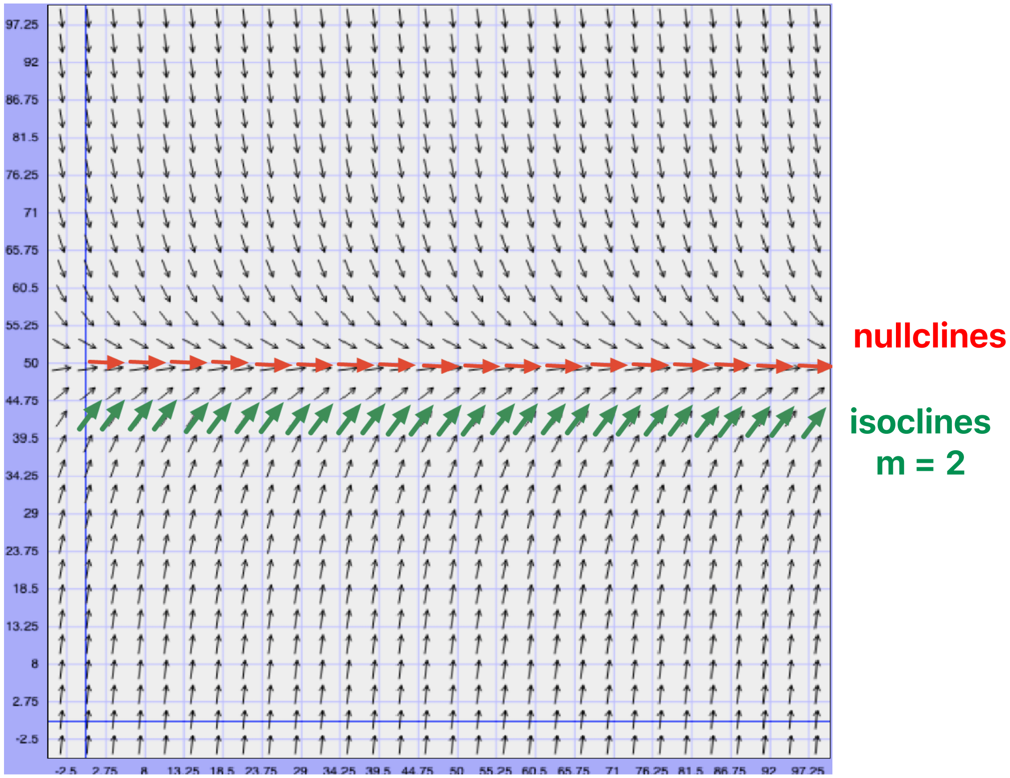
$$m\frac{dv}{dt} = F = -9.8m + \gamma v$$

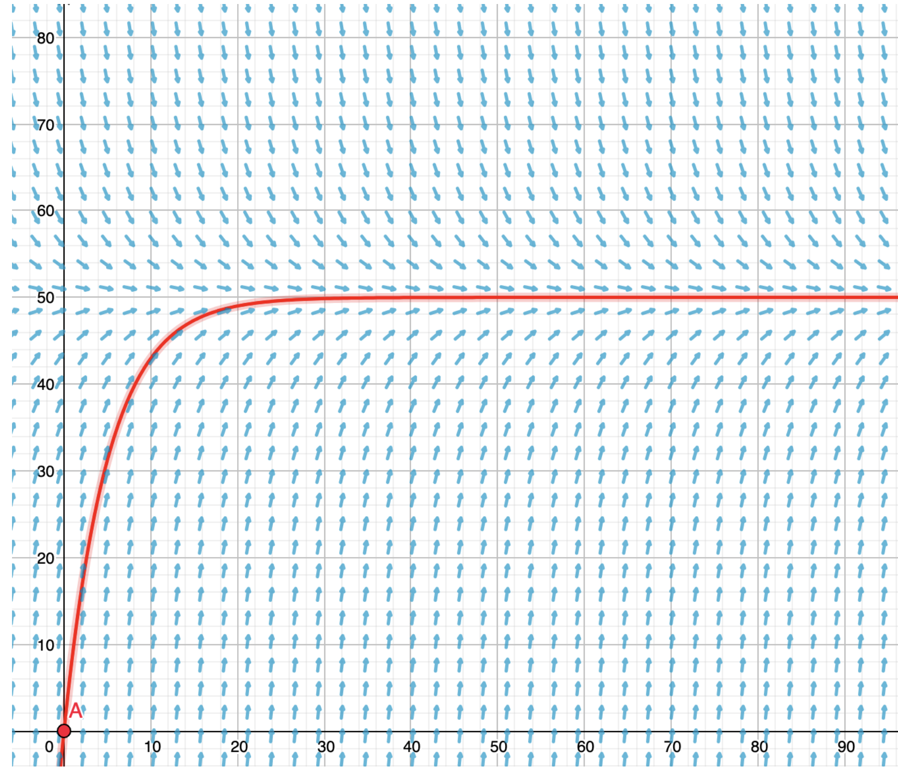
$$\frac{dv}{dt} = -9.8 + \kappa v \qquad \text{where} \quad \kappa = \frac{\gamma}{m}$$

So, let's assume that we have a mass of 2 kg and that $\gamma = 0.392$, then we have

$$\frac{dv}{dt} = -9.8 + 0.196v$$

Let us now pick different values of v and compute the slope of the tangent line for those values of the velocity.


nullclines
isoclines
m = 2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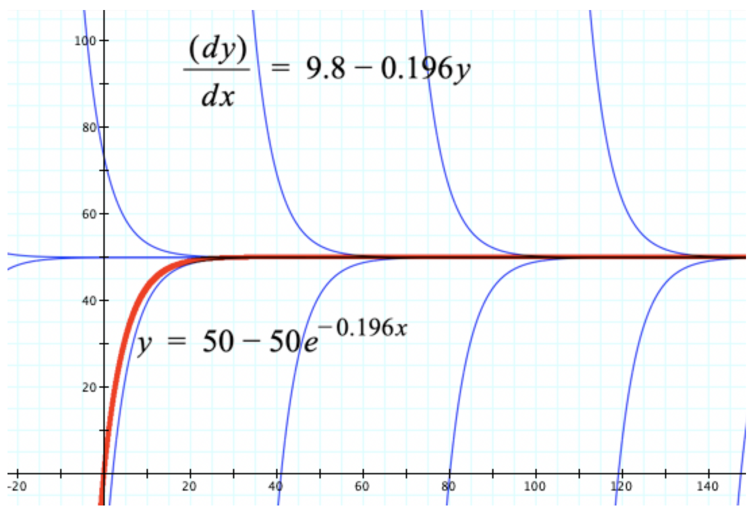
$\frac{(dy)}{dx} = 9.8 - 0.196y$
$y = 50 - 50e^{-0.196x}$