Sample Problems

Compute each of the following integrals. Assume that a and b are positive numbers.

1. $\int \sin x \; dx$
2. $\int \cos 5x \; dx$
3. $\int \cos x \sin^4 x \; dx$
4. $\int \csc^2 x \; dx$
5. $\int \tan x \; dx$
6. $\int \cot x \; dx$
7. $\int \sec x \; dx$
8. $\int \csc x \; dx$
9. $\int \sin^2 x \; dx$
10. $\int \sin^3 x \; dx$
11. $\int \sin^4 x \; dx$
12. $\int \sin^5 x \; dx$
13. $\int \frac{1}{a^2 + b^2x^2} \; dx$
14. $\int \frac{1}{\sqrt{a^2 - x^2}} \; dx$
15. $\int \frac{\sec\left(\sqrt{x}\right)}{\sqrt{x}} \; dx$
16. $\int_0^{\pi/3} \sqrt{1 + \cos 2x} \; dx$
17. $\int_0^{\pi/2} \sqrt{1 - \cos x} \; dx$
18. $\int \tan^3 x \; dx$
19. $\int \sin 7x \cos 3x \; dx$
20. $\int \sin 10x \sin 4x \; dx$

Practice Problems

1. $\int \cos 3x \; dx$
2. $\int \sin\left(4x - \frac{\pi}{5}\right) \; dx$
3. $\int \sec\theta \tan\theta \; d\theta$
4. $\int \sec^2 \theta \; d\theta$
5. $\int x \tan\left(x^2\right) \; dx$
6. $\int \cot\left(2x - \pi\right) \; dx$
7. $\int \cos^2 x \; dx$
8. $\int \cos^2 (2x) \; dx$
9. $\int \cos^3 x \; dx$
10. $\int \cos^4 x \; dx$
11. $\int \cos^5 x \; dx$
12. $\int \sin x \cos^5 x \; dx$
13. $\int \sin^3 x \cos^5 x \; dx$
14. $\int \tan^2 x \; dx$
15. $\int_0^{\pi/6} \sqrt{1 - \cos 6x} \; dx$
16. $\int \sin 2a \cos 8a \; da$
17. $\int \cos b \cos 11b \; db$
18. $\int \sin 6\theta \sin 14\theta \; d\theta$
19. $\int \cos 11m \sin 3m \; dm$

Sample Problems - Answers

1.) $-\cos x + C$ 2.) $\frac{1}{5}\sin 5x + C$ 3.) $\frac{1}{5}\sin^5 x + C$ 4.) $-\cot x + C$ 5.) $-\ln|\cos x| + C = \ln|\sec x| + C$

6.) $\ln|\sin x| + C$ 7.) $\ln|\sec x + \tan x| + C$ 8.) $-\ln|\csc x + \cot x| + C$ 9.) $\frac{1}{2}x - \frac{1}{4}\sin 2x + C$

10.) $\frac{1}{3}\cos^3 x - \cos x + C$ 11.) $-\frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + \frac{3}{8}x + C$ 12.) $-\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C$

13.) $\frac{1}{ab}\tan^{-1}\left(\frac{b}{a}x\right) + C$ 14.) $\sin^{-1}\left(\frac{x}{a}\right) + C$ 15.) $2\ln\left|\sec\left(\sqrt{x}\right) + \tan\left(\sqrt{x}\right)\right| + C$ 16.) $\frac{\sqrt{6}}{2}$

17.) $2\sqrt{2} - 2$ 18.) $\frac{1}{2}\sec^2 x + \ln|\cos x| + C$ 19.) $-\frac{1}{20}\cos 10x - \frac{1}{8}\cos 4x + C$ 20.) $\frac{1}{12}\sin 6x - \frac{1}{28}\sin 14x + C$

Practice Problems - Answers

1.) $\frac{1}{3}\sin 3x + C$ 2.) $-\frac{1}{4}\cos\left(4x - \frac{\pi}{5}\right) + C$ 3.) $\sec\theta + C$ 4.) $\tan\theta + C$ 5.) $\frac{1}{2}\ln\left|\sec\left(x^2\right)\right| + C$

6.) $\frac{1}{2}\ln|\sin(2x - \pi)| + C$ 7.) $\frac{1}{2}x + \frac{1}{4}\sin 2x + C$ 8.) $\frac{1}{2}x + \frac{1}{8}\sin 4x + C$ 9.) $\sin x - \frac{1}{3}\sin^3 x + C$

10.) $\frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$ 11.) $\frac{1}{5}\sin^5 x - \frac{2}{3}\sin^3 x + \sin x + C$ 12.) $-\frac{1}{6}\cos^6 x + C$

13.) $-\frac{1}{6}\cos^6 x + \frac{1}{8}\cos^8 x + C$ 14.) $-x + \tan x + C$ 15.) $\frac{\sqrt{2}}{3}$ 16.) $\frac{1}{12}\cos 6a - \frac{1}{20}\cos 10a + C$

17.) $\frac{1}{20}\sin 10b + \frac{1}{24}\sin 12b + C$ 18.) $\frac{1}{16}\sin 8\theta - \frac{1}{40}\sin 20\theta + C$ 19.) $\frac{1}{16}\cos 8m - \frac{1}{28}\cos 14m + C$

Sample Problems - Solutions

1. $\displaystyle\int \sin x \; dx$

   Solution: This is a basic integral we know from differentiating basic trigonometric functions. Since $\dfrac{d}{dx}\cos x = -\sin x$, clearly $\dfrac{d}{dx}(-\cos x) = \sin x$ and so $\displaystyle\int \sin x \; dx = \boxed{-\cos x + C}$.

2. $\displaystyle\int \cos 5x \; dx$

   Solution: We know that $\dfrac{d}{dx}\cos x = -\sin x + C$. We will use substitution. Let $u = 5x$ and then $du = 5dx$ and so $\dfrac{du}{5} = dx$.

$$\int \cos 5x \; dx = \int \cos u \left(\frac{du}{5}\right) = \frac{1}{5}\int \cos u \; du = \boxed{\frac{1}{5}\sin 5x + C}$$

   Note: Once we have enough practice, there is no need to perform this substitution in writing. We can just simply write $\displaystyle\int \cos 5x \; dx = \frac{1}{5}\sin 5x + C$.

3. $\displaystyle\int \cos x \sin^4 x \; dx$

   Solution: Let $u = \sin x$. Then $du = \cos x dx$.

$$\int \cos x \sin^4 x \; dx = \int \sin^4 x \;\; (\cos x dx) = \int u^4 \;\; u = \frac{1}{5}u^5 + C = \boxed{\frac{1}{5}\sin^5 x + C}$$

4. $\displaystyle\int \csc^2 x \; dx$

   Solution: We need to remember that $\dfrac{d}{dx}\cot x = -\csc^2 x$.

$$\int \csc^2 x \; dx = -\int -\csc^2 x \; dx = \boxed{-\cot x + C}$$

5. $\displaystyle\int \tan x \; dx$

   Solution: Let $u = \cos x$. Then $du = -\sin x \; dx$.

$$\begin{aligned}\int \tan x \; dx &= \int \frac{\sin x}{\cos x} \; dx = \int \frac{1}{u} \;\; (\sin x du) = \int \frac{1}{u} \;\; (-du) = -\int \frac{1}{u} \; du = -\ln|u| + C = -\ln|\cos x| + C \\ &= \ln\left|(\cos x)^{-1}\right| + C = \boxed{\ln|\sec x| + C}\end{aligned}$$

6. $\displaystyle\int \cot x \; dx$

   Solution: Let $u = \sin x$. Then $du = \cos x \; dx$.

$$\int \cot x \; dx = \int \frac{\cos x}{\sin x} \; dx = \int \frac{1}{u} \;\; (\cos x du) = \int \frac{1}{u} \; du = \ln|u| + C = \boxed{\ln|\sin x| + C}$$

7. $\displaystyle\int \sec x \; dx$

Solution: $\displaystyle\int \sec x \; dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} \; dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \; dx$

From here we will use substitution. Recall that $\dfrac{d}{dx} \sec x = \sec x \tan x$ and $\dfrac{d}{dx} \tan x = \sec^2 x$. Let $u = \sec x + \tan x$. Then $du = \left(\sec x \tan x + \sec^2 x\right) dx$.

$$\int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \; dx = \int \frac{1}{u} \; \left(\sec^2 x + \sec x \tan x\right) dx = \int \frac{1}{u} \; du = \ln |u| + C = \boxed{\ln |\sec x + \tan x| + C}$$

8. $\displaystyle\int \csc x \; dx$

Solution: $\displaystyle\int \csc x \; dx = \int \csc x \cdot \frac{\csc x + \cot x}{\csc x + \cot x} \; dx = \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x} \; dx$

From here we will use substitution. Recall that $\dfrac{d}{dx} \csc x = -\csc x \cot x$ and $\dfrac{d}{dx} \cot x = -\csc^2 x$. Let $u = \csc x + \cot x$. Then $du = \left(-\csc^2 x - \csc x \cot x\right) dx$.

$$\begin{aligned} \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x} \; dx &= \int \frac{1}{u} \; \left(\csc^2 x + \csc x \cot x\right) dx = \int \frac{1}{u} \; (-du) = -\int \frac{1}{u} \; du = -\ln |u| + C \\ &= \boxed{-\ln |\csc x + \cot x| + C} \end{aligned}$$

9. $\displaystyle\int \sin^2 x \; dx$

Solution: Recall the double angle formula for cosine, $\cos 2x = 1 - 2\sin^2 x$. We solve this for $\sin^2 x$

$$\sin^2 x = \frac{1}{2}\left(1 - \cos 2x\right)$$

$$\begin{aligned} \int \sin^2 x \; dx &= \int \frac{1}{2}\left(1 - \cos 2x\right) \; dx = \frac{1}{2}\left(\int 1 \; dx - \int \cos 2x \; dx\right) = \frac{1}{2}\left(x + C_1 - \frac{1}{2}\sin 2x + C_2\right) \\ &= \boxed{\frac{1}{2}x - \frac{1}{4}\sin 2x + C} \end{aligned}$$

10. $\displaystyle\int \sin^3 x \; dx$

Solution:

$$\int \sin^3 x \; dx = \int \sin x \sin^2 x \; dx = \int \sin x \left(1 - \cos^2 x\right) \; dx$$

Let $u = \cos x$. Then $du = -\sin x dx$

$$\begin{aligned} \int \sin^3 x \; dx &= \int \sin x \left(1 - \cos^2 x\right) \; dx = \int \left(1 - \cos^2 x\right)\left(\sin x dx\right) = \int \left(1 - u^2\right)(-du) = \int \left(u^2 - 1\right) du \\ &= \frac{1}{3}u^3 - u + C = \boxed{\frac{1}{3}\cos^3 x - \cos x + C} \end{aligned}$$

11. $\int \sin^4 x \, dx$

Solution: We use the double angle formula for cosine to express $\sin^2 x$.

$$\cos 2x = 1 - 2\sin^2 x \quad \Longrightarrow \quad \sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\int \sin^4 x \, dx = \int \left(\sin^2 x\right)^2 \, dx = \int \left(\frac{1}{2}(1-\cos 2x)\right)^2 \, dx = \frac{1}{4}\int (1-\cos 2x)^2 \, dx = \frac{1}{4}\int \left(1 - 2\cos 2x + \cos^2 2x\right) \, dx$$

We use the double angle formula for cosine again to express $\cos^2 2x$.

$$\cos 4x = 2\cos^2 2x - 1 \quad \Longrightarrow \quad \cos^2 2x = \frac{1}{2}(\cos 4x + 1)$$

$$\begin{aligned}
\int \sin^4 x \, dx &= \frac{1}{4}\int \left(1 - 2\cos 2x + \cos^2 2x\right) \, dx = \frac{1}{4}\int \left(1 - 2\cos 2x + \frac{1}{2}(\cos 4x + 1)\right) \, dx \\
&= \int \left(\frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x + \frac{1}{8}\right) \, dx = \int \left(-\frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x + \frac{3}{8}\right) \, dx \\
&= -\frac{1}{2}\left(\frac{1}{2}\right)\sin 2x + \frac{1}{8}\left(\frac{1}{4}\right)\sin 4x + \frac{3}{8}x + C = \boxed{-\frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + \frac{3}{8}x + C}
\end{aligned}$$

12. $\int \sin^5 x \, dx$

Solution: This method works with odd powers of $\sin x$ or $\cos x$. We will separate one factor of $\sin x$ from the rest which will be expressed in terms of $\cos x$.

$$\begin{aligned}
\int \sin^5 x \, dx &= \int \sin x \sin^4 x \, dx = \int \sin x \sin^4 x \, dx = \int \sin x \left(\sin^2 x\right)^2 \, dx = \int \sin x \left(1 - \cos^2 x\right)^2 \, dx \\
&= \int \sin x \left(1 - 2\cos^2 x + \cos^4 x\right) \, dx
\end{aligned}$$

We proceed with substitution. Let $u = \cos x$. Then $du = -\sin x dx$.

$$\begin{aligned}
\int \sin^5 x \, dx &= \int \sin x \left(1 - 2\cos^2 x + \cos^4 x\right) \, dx = \int \left(1 - 2\cos^2 x + \cos^4 x\right) \, (\sin x dx) \\
&= \int \left(1 - 2u^2 + u^4\right) \, (-du) = \int \left(-1 + 2u^2 - u^4\right) \, du = -u + \frac{2}{3}u^3 - \frac{1}{5}u^5 + C \\
&= \boxed{-\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C}
\end{aligned}$$

13. $\int \frac{1}{a^2 + b^2x^2} \, dx$

Solution: The basic integral here is $\int \frac{1}{x^2+1} \, dx = \tan^{-1} x + C$. We need a substitution under which $a^2x^2 = b^2u^2$. This would be convenient because then

$$\frac{1}{a^2x^2 + b^2} = \frac{1}{b^2u^2 + b^2} = \frac{1}{b^2} \cdot \frac{1}{u^2+1}$$

So we will pursue this substitution. We solve $a^2x^2 = b^2u^2$ for a possible value of u and obtain $u = \dfrac{a}{b}x$. Then $du = \dfrac{a}{b}dx$ and so $\dfrac{b}{a}du = dx$.

$$\begin{aligned}\int \frac{1}{a^2x^2+b^2}\,dx &= \int \frac{1}{b^2u^2+b^2}\left(\frac{b}{a}du\right) = \int \frac{1}{b^2}\cdot\frac{1}{u^2+1}\cdot\frac{b}{a}\,du = \frac{b}{ab^2}\int\frac{1}{u^2+1}\,du = \frac{1}{ab}\tan^{-1}u + C\\ &= \boxed{\frac{1}{ab}\tan^{-1}\left(\frac{b}{a}x\right)+C}\end{aligned}$$

14. $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\,dx$

Solution: The basic integral here is $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$. We need a substitution under which $x^2 = a^2u^2$. This would be useful because then

$$\frac{1}{\sqrt{a^2-x^2}} = \frac{1}{\sqrt{a^2-a^2u^2}} = \frac{1}{\sqrt{a^2\left(1-u^2\right)}} = \frac{1}{a\sqrt{1-u^2}}$$

So we will pursue this substitution. We solve $x^2 = a^2u^2$ for a possible value of u and obtain $x = au$ and $dx = adu$.

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \int \frac{1}{\sqrt{a^2-a^2u^2}}\,(adu) = \int \frac{a}{a\sqrt{1-u^2}}\,du = \int \frac{1}{\sqrt{1-u^2}}\,du- = \sin^{-1}u + C = \boxed{\sin^{-1}\left(\frac{x}{a}\right)+C}$$

15. $\displaystyle\int \frac{\sec\left(\sqrt{x}\right)}{\sqrt{x}}\,dx$

Let $u = \sqrt{x}$. Then $du = \dfrac{1}{2\sqrt{x}}dx$.

$$\begin{aligned}\int \frac{\sec\left(\sqrt{x}\right)}{\sqrt{x}}\,dx &= 2\int \frac{\sec\left(\sqrt{x}\right)}{2\sqrt{x}}\,dx = 2\int \sec\left(\sqrt{x}\right)\left(\frac{1}{2\sqrt{x}}dx\right) = 2\int \sec u\;du = 2\ln\left|\sec u + \tan u\right| + C\\ &= \boxed{2\ln\left|\sec\left(\sqrt{x}\right)+\tan\left(\sqrt{x}\right)\right| + C}\end{aligned}$$

16. $\displaystyle\int_0^{\pi/3} \sqrt{1+\cos 2x}\;dx$

Solution: We will yet again use the double angle formula for cosine, this time to eliminate the square root.

$$\cos 2x = 2\cos^2 x - 1 \quad\Longrightarrow\quad 2\cos^2 x = \cos 2x + 1$$

$$\int_0^{\pi/3} \sqrt{1+\cos 2x}\;dx = \int_0^{\pi/3}\sqrt{2\cos^2 x}\;dx = \sqrt{2}\int_0^{\pi/3}\sqrt{\cos^2 x}\;dx = \sqrt{2}\int_0^{\pi/3}\left|\cos x\right|\;dx$$

Since $f(x) = \cos x$ is positive on $\left[0, \dfrac{\pi}{3}\right]$, we can simplify $\left|\cos x\right| = \cos x$

$$\sqrt{2}\int_0^{\pi/3}\left|\cos x\right|\;dx = \sqrt{2}\int_0^{\pi/3}\cos x\;dx = \sqrt{2}\left(\sin x\Big|_0^{\pi/3}\right) = \sqrt{2}\left(\sin\frac{\pi}{3} - \sin 0\right) = \sqrt{2}\left(\frac{\sqrt{3}}{2} - 0\right) = \boxed{\frac{\sqrt{6}}{2}}$$

17. $\displaystyle\int_0^{\pi/2} \sqrt{1-\cos x}\ dx$

Solution:

$$\cos 2\theta = 1 - 2\sin^2\theta \qquad \Longrightarrow \qquad 2\sin^2\theta = 1-\cos 2\theta$$

We substitute $\theta = \dfrac{x}{2}$ into this and obtain

$$2\sin^2\frac{x}{2} = 1-\cos x$$

$$\int_0^{\pi/2} \sqrt{1-\cos x}\ dx = \int_0^{\pi/2} \sqrt{2\sin^2\frac{x}{2}}\ dx = \sqrt{2}\int_0^{\pi/2} \left|\sin\frac{x}{2}\right|\ dx$$

Since $f(x) = \sin\dfrac{x}{2}$ is non-negative on $\left[0, \dfrac{\pi}{2}\right]$, we can simplify $\left|\sin\dfrac{x}{2}\right| = \sin\dfrac{x}{2}$

$$\begin{aligned}\sqrt{2}\int_0^{\pi/2} \sin\frac{x}{2}\ dx &= \sqrt{2}\left(-2\cos\frac{x}{2}\bigg|_0^{\pi/2}\right) = -2\sqrt{2}\left(\cos\frac{x}{2}\bigg|_0^{\pi/2}\right) = -2\sqrt{2}\left(\cos\frac{\pi}{4}-\cos 0\right)\\ &= -2\sqrt{2}\left(\frac{\sqrt{2}}{2}-1\right) = -2+2\sqrt{2} = \boxed{2\sqrt{2}-2}\end{aligned}$$

18. $\displaystyle\int \tan^3 x\ dx$

Solution: Let $u = \cos x$. Then $du = -\sin x dx$

$$\begin{aligned}\int \tan^3 x\ dx &= \int \frac{\sin^3 x}{\cos^3 x}\ dx = \int \sin x\frac{\sin^2 x}{\cos^3 x}\ dx = \int \frac{1-\cos^2 x}{\cos^3 x}\ \sin x dx = \int \frac{1-u^2}{u^3}\ (-du) = \int \frac{u^2-1}{u^3}\ du\\ &= \int \frac{u^2}{u^3} - \frac{1}{u^3}\ du = \int \frac{1}{u} - u^{-3}\ du = \ln|u| - \frac{u^{-2}}{-2} + C = \ln|u| + \frac{1}{2u^2} + C\\ &= \ln|\cos x| + \frac{1}{2}\sec^2 x + C\end{aligned}$$

19. $\displaystyle\int \sin 7x\cos 3x\ dx$

Solution: We wil use the product-to-sum identities to trasform this product into a sum. We write the sine formula for the sum and the difference of these two angles.

$$\begin{aligned}\sin 10x &= \sin(7x+3x) = \sin 7x\cos 3x + \cos 7x\sin 3x\\ \sin 4x &= \sin(7x-3x) = \sin 7x\cos 3x - \cos 7x\sin 3x\end{aligned}$$

We will add the two equations

$$\begin{aligned}\sin 10x + \sin 4x &= 2\sin 7x\cos 3x\\ \frac{1}{2}(\sin 10x + \sin 4x) &= \sin 7x\cos 3x\end{aligned}$$

We can very easily integrate $\dfrac{1}{2}(\sin 10x + \sin 4x)$

$$\begin{aligned}\int \sin 7x\cos 3x\ dx &= \int \frac{1}{2}(\sin 10x + \sin 4x)\ dx = \frac{1}{2}\int \sin 10x + \sin 4x\ dx\\ &= \frac{1}{2}\left(\frac{1}{10}\right)(-\cos 10x) + \frac{1}{2}\left(\frac{1}{4}(-\cos 4x)\right) + C = \boxed{-\frac{1}{20}\cos 10x - \frac{1}{8}\cos 4x + C}\end{aligned}$$

20. $\displaystyle\int \sin 10x \sin 4x \; dx$

Solution: We wil use the product-to-sum identities to trasform this product into a sum. We write the cosine formula for the sum and the difference of these two angles.

$$\begin{aligned}\cos 14x &= \cos(10x+4x) = \cos 10x \cos 4x - \sin 10x \sin 4x \\ \cos 6x &= \cos(10x-4x) = \cos 10x \cos 4x + \sin 10x \sin 4x\end{aligned}$$

We will subtract the first equation from the first one

$$\begin{aligned}\cos 6x - \cos 14x &= 2\sin 10x \sin 4x \\ \frac{1}{2}(\cos 6x - \cos 14x) &= \sin 10x \sin 4x\end{aligned}$$

We can very easily integrate $\frac{1}{2}(\cos 6x - \cos 14x)$

$$\begin{aligned}\int \sin 10x \sin 4x \; dx &= \int \frac{1}{2}(\cos 6x - \cos 14x) \; dx = \frac{1}{2}\int \cos 6x - \cos 14x \; dx \\ &= \frac{1}{2}\left(\frac{1}{6}\right)(\sin 6x) - \frac{1}{2}\left(\frac{1}{14}(\sin 14x)\right) + C = \boxed{\frac{1}{12}\sin 6x - \frac{1}{28}\sin 14x + C}\end{aligned}$$

For more documents like this, visit our page at http://www.teaching.martahidegkuti.com and click on Lecture Notes. E-mail questions or comments to mhidegkuti@ccc.edu.

Sample Problems

Compute each of the following integrals.

1. $\displaystyle\int \frac{1}{\sqrt{x^2+4}}\,dx$
2. $\displaystyle\int \sqrt{1-x^2}\,dx$
3. $\displaystyle\int \frac{1}{\sqrt{x^2-9}}\,dx$
4. $\displaystyle\int \frac{x^2}{\sqrt{16-x^2}}\,dx$
5. $\displaystyle\int \sqrt{x^2+4}\,dx$
6. $\displaystyle\int \frac{x^2}{\sqrt{x^2+9}}\,dx$
7. $\displaystyle\int \frac{x^6}{\sqrt{1-x^{14}}}\,dx$
8. $\displaystyle\int_0^1 \frac{\tan^{-1} x}{x^2+1}\,dx$

Practice Problems

Compute each of the following integrals. Please note that some of the integrals can also be solved using other, previously seen methods.

1. $\displaystyle\int \frac{1}{\sqrt{x^2-25}}\,dx$
2. $\displaystyle\int \frac{x}{\sqrt{x^2-25}}\,dx$
3. $\displaystyle\int \frac{x^2}{\sqrt{x^2-25}}\,dx$
4. $\displaystyle\int \sqrt{x^2-25}\,dx$
5. $\displaystyle\int \sqrt{16-x^2}\,dx$
6. $\displaystyle\int \frac{1}{\sqrt{16-x^2}}\,dx$
7. $\displaystyle\int \frac{x}{\sqrt{16-x^2}}\,dx$
8. $\displaystyle\int \frac{x^2}{\sqrt{16-x^2}}\,dx$
9. $\displaystyle\int \sqrt{x^2+1}\,dx$
10. $\displaystyle\int \frac{1}{\sqrt{x^2+1}}\,dx$
11. $\displaystyle\int \frac{x}{\sqrt{x^2+1}}\,dx$
12. $\displaystyle\int \frac{x^2}{\sqrt{x^2+1}}\,dx$

Sample Problems - Answers

1.) $\ln\left|x+\sqrt{x^2+4}\right|+C$ 2.) $\frac{1}{2}\sin^{-1}x+\frac{1}{2}x\sqrt{1-x^2}+C$ 3.) $\ln\left|x+\sqrt{x^2-9}\right|+C$

4.) $8\sin^{-1}\left(\frac{x}{4}\right)-\frac{1}{2}x\sqrt{16-x^2}+C$ 5.) $\frac{1}{2}x\sqrt{x^2+4}+2\ln\left|x+\sqrt{x^2+4}\right|+C$

6.) $\frac{1}{2}x\sqrt{x^2+9}-\frac{9}{2}\ln\left|x+\sqrt{x^2+9}\right|+C$ 7.) $\frac{1}{7}\sin^{-1}\left(x^7\right)+C$ 8.) $\frac{\pi^2}{32}$

Practice Problems - Answers

1.) $\ln\left|x+\sqrt{x^2-25}\right|+C$ 2.) $\sqrt{x^2-25}+C$ 3.) $\frac{1}{2}x\sqrt{x^2-25}+\frac{25}{2}\ln\left|x+\sqrt{x^2-25}\right|+C$

4.) $\frac{1}{2}x\sqrt{x^2-25}-\frac{25}{2}\ln\left|x+\sqrt{x^2-25}\right|+C$ 5.) $8\sin^{-1}\left(\frac{x}{4}\right)+\frac{1}{2}x\sqrt{16-x^2}+C$ 6.) $\sin^{-1}\left(\frac{x}{4}\right)+C$

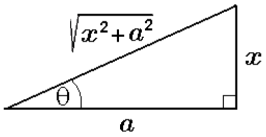
7.) $-\sqrt{16-x^2}+C$ 8.) $8\sin^{-1}\left(\frac{x}{4}\right)-\frac{1}{2}x\sqrt{16-x^2}+C$ 9.) $\frac{1}{2}x\sqrt{x^2+1}+\frac{1}{2}\ln\left|x+\sqrt{x^2+1}\right|+C$

10.) $\ln\left|x+\sqrt{x^2+1}\right|+C$ 11.) $\sqrt{x^2+1}+C$ 12.) $\frac{1}{2}x\sqrt{x^2+1}-\frac{1}{2}\ln\left|x+\sqrt{x^2+1}\right|+C$

Sample Problems - Solutions

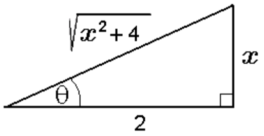
Trigonometric substitution is a technique of integration. It is especially useful in handling expressions under a square root sign.

Case 1. The substitution $x = a\tan\theta$. This is useful in handling an integral involving $\sqrt{x^2+a^2}$.
Let $x = a\tan\theta$ where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. (That is the same thing as stating that $\theta = \tan^{-1}\frac{x}{a}$. The interval between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ is the domain of the inverse function $\tan^{-1} x$.)
The picture below shows the reference triangle we use for this substitution.



Using this triangle, we do not have to do heavy duty algebra because we can read (up to sign) the trigonometric functions of θ in terms of x and a.

Example 1: Compute the integral $\int \frac{1}{\sqrt{x^2+4}}\,dx$.
Solution: We will use a trigonometric substitution. We start with a reference triangle where the hypotenuse is the denominator. Using the substitution $x = 2\tan\theta$, (where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$) we will transform the integral into one in θ.



From the triangle, $x = 2\tan\theta$. Then $dx = 2\sec^2\theta\,d\theta$. The expression $\sqrt{x^2+4}$ becomes $2\sec\theta$ - using the picture, or using algebra. Recall the identity $\tan^2 x + 1 = \sec^2 x$

$$\sqrt{x^2+4} = \sqrt{(2\tan\theta)^2+4} = \sqrt{4\tan^2\theta+4} = \sqrt{4}\sqrt{\tan^2\theta+1} = 2\sqrt{\sec^2\theta} = 2\left|\sec\theta\right|$$

Because θ is in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\sec x$ is positive and so $|\sec x| = \sec x$.

$$\int \frac{1}{\sqrt{x^2+4}}\,dx = \int \frac{1}{2\sec\theta}\left(2\sec^2\theta\,d\theta\right) = \int \sec\theta\,d\theta$$

This is an integral we have already seen: we can either use substitution (see in that handout) or partial fraction(see in that handout). Either way,

$$\int \sec\theta\,d\theta = \ln\left|\sec\theta+\tan\theta\right| + C$$

Now we need to reverse the substitution and write the result as an expression of x. This is where the reference triangle comes handy.

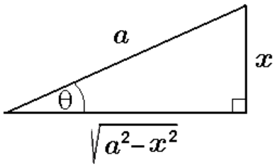
$$\sec\theta = \frac{\sqrt{x^2+4}}{2} \quad \text{and} \quad \tan\theta = \frac{x}{2}$$

Thus the answer is $\int \frac{1}{\sqrt{x^2+4}}\,dx = \ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right| + C$. This expression can be further simplified:

$$\ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right| + C = \ln\left|\frac{\sqrt{x^2+4}+x}{2}\right| + C = \ln\left|\sqrt{x^2+4}+x\right| - \ln 2 + C = \ln\left|\sqrt{x^2+4}+x\right| + C$$

and so the final answer is $\boxed{\ln\left|\sqrt{x^2+4}+x\right|+C}$.

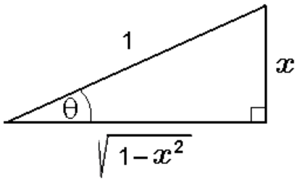
Case 2. The substitution $x = a\sin\theta$ where $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$. This is useful in handling an integral involving $\sqrt{a^2-x^2}$. The picture below shows the reference triangle we use for this substitution.



Using this triangle, we can read (up to sign) the trigonometric functions of θ in terms of x and a.

Example 2: Compute the integral $\displaystyle\int \sqrt{1-x^2}\ dx$.

Solution: This is a very famous integral because it leads to the area formula of the unit circle. We will use a trigonometric substitution. We start with a reference triangle where the $\sqrt{1-x^2}$ is one of the legs. Using the substitution $x = \sin\theta$, ($-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$) we will transform the integral into one in θ.



From the triangle, $x = \sin\theta$. Then $dx = \cos\theta\ d\theta$. The expression $\sqrt{1-x^2}$ becomes

$$\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = |\cos\theta|$$

Because θ is in the interval $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$, $\cos x$ is positive and so $|\cos x| = \cos x$.

$$\int \sqrt{1-x^2}\ dx = \int \cos\theta\ \ (\cos\theta\ d\theta) = \int \cos^2\theta\ d\theta$$

This is an integral we have already seen; we can simplify it using the double angle formula for cosine.

$$\cos 2\theta = 2\cos^2\theta - 1 \qquad \Longrightarrow \qquad \cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$$

$$\begin{aligned}\int \cos^2\theta\ d\theta &= \int \frac{1}{2}(\cos 2\theta + 1)\ d\theta = \frac{1}{2}\int \cos 2\theta + 1\ d\theta = \frac{1}{2}\left(\frac{1}{2}\sin 2\theta + \theta\right) + C \\ &= \frac{1}{2}\left(\frac{1}{2}(2\sin\theta\cos\theta) + \theta\right) + C = \frac{1}{2}\sin\theta\cos\theta + \frac{1}{2}\theta + C\end{aligned}$$

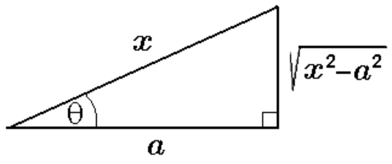
Now we need to reverse the substitution and write the result as an expression of x. This is where the reference triangle comes handy.

$$\sin\theta = x, \quad \cos\theta = \sqrt{1-x^2} \quad \text{and} \quad \theta = \sin^{-1} x$$

Thus the answer is $\displaystyle\int \sqrt{1-x^2}\ dx = \frac{1}{2}\sin\theta\cos\theta + \frac{1}{2}\theta + C = \boxed{\frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x + C}$

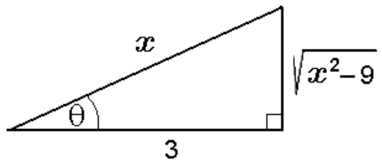
Note that if we now compute $\displaystyle\int\limits_{-1}^{1} \sqrt{1-x^2}\ dx$ the result is the area of the unit semi-circle, $\dfrac{\pi}{2}$.

Case 3. The substitution $x = a\sec\theta$ where $0 < \theta < \dfrac{\pi}{2}$. This is useful in handling an integral involving $\sqrt{x^2 - a^2}$. The picture below shows the reference triangle we use for this substitution.



Using this triangle, we can read (up to sign) the trigonometric functions of θ in terms of x and a.

Example 3: Compute the integral $\displaystyle\int \frac{1}{\sqrt{x^2-9}}\, dx$.
Solution: We will use a trigonometric substitution. We start with a reference triangle where the hypotenuse is x and one shorter side is 3. Using the substitution $x = 3\sec\theta$, we will transform the integral into one in θ.



From the triangle, $x = 3\sec\theta$. Then $dx = 3\sec\theta\tan\theta\ d\theta$. The expression $\sqrt{x^2-9}$ becomes $3\tan\theta$ - either from the picture or using algebra. Recall the identity $\sec^2 x = \tan^2 x + 1$

$$\sqrt{x^2-9} = \sqrt{(3\sec\theta)^2 - 9} = \sqrt{9\sec^2\theta - 9} = \sqrt{9}\sqrt{\sec^2\theta - 1} = 3\sqrt{\tan^2\theta} = 3\left|\tan\theta\right|$$

Because $0 < \theta < \dfrac{\pi}{2}$, $\tan\theta$ is positive and so $\left|\tan\theta\right| = \tan\theta$.

$$\int \frac{1}{\sqrt{x^2-9}}\, dx = \int \frac{1}{3\tan\theta}\ (3\sec\theta\tan\theta\ d\theta) = \int \sec\theta\ d\theta$$

Again,

$$\int \sec\theta\ d\theta = \ln\left|\sec\theta + \tan\theta\right| + C$$

Now we need to reverse the substitution and write the result as an expression of x. This is where the reference triangle comes handy.

$$\sec\theta = \frac{x}{3} \qquad \text{and} \qquad \tan\theta = \frac{\sqrt{x^2-9}}{3}$$

Thus the answer is $\displaystyle\int \frac{1}{\sqrt{x^2-9}}\, dx = \ln\left|\sec\theta + \tan\theta\right| + C = \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| + C$. We can still simplify this result a bit:

$$\ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| + C = \ln\left|\frac{x+\sqrt{x^2-9}}{3}\right| + C = \ln\left|x + \sqrt{x^2-9}\right| - \ln 3 + C = \ln\left|x + \sqrt{x^2-9}\right| + C_2$$

Thus the final answer is $\displaystyle\int \frac{1}{\sqrt{x^2-9}}\, dx = \boxed{\ln\left|x + \sqrt{x^2-9}\right| + C}$.

Example 4: Compute the integral $\displaystyle\int \frac{x^2}{\sqrt{16-x^2}}\,dx$

Solution: Let $x = 4\sin\theta$ where $-\dfrac{\pi}{2} \leq \theta \leq \dfrac{\pi}{2}$. Then $dx = 4\cos\theta\ d\theta$ and

$$\sqrt{16-x^2} = \sqrt{16-16\sin^2\theta} = \sqrt{16}\sqrt{1-\sin^2\theta} = 4\sqrt{\cos^2\theta} = 4\left|\cos\theta\right|$$

Because $-\dfrac{\pi}{2} \leq \theta \leq \dfrac{\pi}{2}$, $\cos\theta$ is non-negative, and $\left|\cos\theta\right| = \cos\theta$. So the integral is

$$\int \frac{x^2}{\sqrt{16-x^2}}\,dx = \int \frac{16\sin^2\theta}{4\cos\theta}\ (4\cos\theta\ d\theta) = \int 16\sin^2\theta\ d\theta = 16\int \sin^2\theta\ d\theta$$

By the double angle formula for cosine, $\cos 2\theta = 1 - 2\sin^2\theta \quad \Longrightarrow \quad \sin^2\theta = \dfrac{1}{2}\left(1-\cos 2\theta\right)$

$$\begin{aligned}\int \frac{x^2}{\sqrt{16-x^2}}\,dx &= 16\int \sin^2\theta\ d\theta = 16\int \frac{1}{2}\left(1-\cos 2\theta\right)\ d\theta = 8\int 1-\cos 2\theta\ d\theta = 8\left(\theta - \frac{1}{2}\sin 2\theta + C\right)\\ &= 8\theta - 4\sin 2\theta + C\end{aligned}$$

Now we need to reverse the substitution and write the result as an expression of x. This is where the reference triangle comes handy. Recall that $x = 4\sin\theta$ and so

$$\begin{aligned}\theta &= \sin^{-1}\left(\frac{x}{4}\right) \qquad \text{and}\\ \sin 2\theta &= 2\sin\theta\cos\theta = 2\sin\theta\sqrt{1-\sin^2\theta} = 2\left(\frac{x}{4}\right)\sqrt{1-\left(\frac{x}{4}\right)^2} = \frac{x}{2}\sqrt{\frac{1}{16}\left(16-x^2\right)}\\ &= \frac{x}{2}\left(\frac{1}{4}\right)\sqrt{16-x^2} = \frac{1}{8}x\sqrt{16-x^2}\end{aligned}$$

And so the final answer is $\displaystyle\int \frac{x^2}{\sqrt{16-x^2}}\,dx = 8\theta - 4\sin 2\theta + C = \boxed{8\sin^{-1}\left(\frac{x}{4}\right) - \frac{1}{2}x\sqrt{16-x^2} + C}$

Example 5: Compute the integral $\displaystyle\int \sqrt{x^2+4}\ dx$

Solution: Let $x = 2\tan\beta$ where $-\dfrac{\pi}{2} \leq \beta \leq \dfrac{\pi}{2}$. Then $dx = 2\sec^2\beta\ d\beta$ and

$$\begin{aligned}\int \sqrt{x^2+4}\ dx &= \int \sqrt{4\tan^2\beta+4}\ \left(2\sec^2\beta\ d\beta\right) = \int 2\sqrt{\tan^2\beta+1}\ \left(2\sec^2\beta\ d\beta\right) = 4\int \left|\sec\beta\right|\ \left(\sec^2\beta\ d\beta\right) =\\ &= 4\int \sec\beta\ \left(\sec^2\beta\ d\beta\right) = 4\int \sec^3\beta\ d\beta\end{aligned}$$

We will compute $\displaystyle\int \sec^3\beta\ d\beta$ by parts.
Let $u = \sec\beta$ and $dv = \sec^2\beta\ d\beta$. Then

$$du = \sec\beta\tan\beta d\beta \quad \text{and} \quad v = \int dv = \int \sec^2\beta\ d\beta$$

In short,

$u = \sec\beta$	$v = \tan\beta$
$du = \sec\beta\tan\beta\ d\beta$	$dv = \sec^2\beta\ d\beta$

$$\begin{aligned}
\int u\ dv &= uv - \int v\ du \text{ becomes} \\
\int \sec\beta\ \sec^2\beta\ d\beta &= \sec\beta\tan\beta - \int \tan\beta\ \sec\beta\tan\beta\ d\beta \\
\int \sec^3\beta\ d\beta &= \sec\beta\tan\beta - \int \tan^2\beta\ \sec\beta\ d\beta \qquad\qquad \text{recall } \tan^2\beta + 1 = \sec^2\beta \\
\int \sec^3\beta\ d\beta &= \sec\beta\tan\beta - \int \left(\sec^2\beta - 1\right)\sec\beta\ d\beta \\
\int \sec^3\beta\ d\beta &= \sec\beta\tan\beta - \int \sec^3\beta - \sec\beta\ d\beta \\
\int \sec^3\beta\ d\beta &= \sec\beta\tan\beta - \int \sec^3\beta\ d\beta + \int \sec\beta\ d\beta \\
2\int \sec^3\beta\ d\beta &= \sec\beta\tan\beta + \int \sec\beta\ d\beta \\
2\int \sec^3\beta\ d\beta &= \sec\beta\tan\beta + \ln|\sec\beta + \tan\beta| + C \\
\int \sec^3\beta\ d\beta &= \frac{1}{2}\sec\beta\tan\beta + \frac{1}{2}\ln|\sec\beta + \tan\beta| + C
\end{aligned}$$

Now the original integral is

$$\begin{aligned}
\int \sqrt{x^2+4}\ dx &= 4\int \sec^3\beta\ d\beta = 4\left(\frac{1}{2}\sec\beta\tan\beta + \frac{1}{2}\ln|\sec\beta+\tan\beta|\right) + C \\
&= 2\sec\beta\tan\beta + 2\ln|\sec\beta+\tan\beta| + C
\end{aligned}$$

Now we need to reverse the substitution and write the result as an expression of x. Recall that $x = 2\tan\beta$. Then $\tan\beta = \dfrac{x}{2}$ and

$$\sec\beta = \sqrt{\tan^2\beta + 1} = \sqrt{\left(\frac{x}{2}\right)^2 + 1} = \sqrt{\frac{1}{4}x^2 + 1} = \sqrt{\frac{1}{4}\left(x^2+4\right)} = \frac{1}{2}\sqrt{x^2+4}$$

and so

$$\begin{aligned}
\int \sqrt{x^2+4}\ dx &= 2\sec\beta\tan\beta + 2\ln|\sec\beta+\tan\beta| + C = 2\left(\frac{1}{2}\sqrt{x^2+4}\right)\left(\frac{x}{2}\right) + 2\ln\left|\frac{1}{2}\sqrt{x^2+4} + \frac{x}{2}\right| + C \\
&= \frac{1}{2}x\sqrt{x^2+4} + 2\ln\left|\frac{x+\sqrt{x^2+4}}{2}\right| + C = \frac{1}{2}x\sqrt{x^2+4} + 2\left(\ln\left|x+\sqrt{x^2+4}\right| - \ln 2\right) + C \\
&= \frac{1}{2}x\sqrt{x^2+4} + 2\ln\left|x+\sqrt{x^2+4}\right| - 2\ln 2 + C = \boxed{\frac{1}{2}x\sqrt{x^2+4} + 2\ln\left|x+\sqrt{x^2+4}\right| + C}
\end{aligned}$$

Example 6: Compute the integral $\displaystyle\int \frac{x^2}{\sqrt{x^2+9}}\,dx$

Let $x = 3\tan\theta$ where $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$. Then $dx = 3\sec^2\theta d\theta$ and

$$\begin{aligned}
\int \frac{x^2}{\sqrt{x^2+9}}\,dx &= \int \frac{9\tan^2\theta}{\sqrt{9\tan^2\theta+9}}\left(3\sec^2\theta\ d\theta\right) = \int \frac{9\tan^2\theta}{3\sqrt{\tan^2\theta+1}}\left(3\sec^2\theta\ d\theta\right) = \int \frac{9\tan^2\theta}{3\left|\sec\theta\right|}\left(3\sec^2\theta\ d\theta\right)\\
&= \int \frac{9\tan^2\theta}{3\sec\theta}\left(3\sec^2\theta\ d\theta\right) = 9\int \tan^2\theta\sec\theta\ d\theta = 9\int\left(\sec^2\theta-1\right)\sec\theta\ d\theta\\
&= 9\int \sec^3\theta-\sec\theta\ d\theta = 9\int\sec^3\theta\ d\theta - 9\int\sec\theta\ d\theta
\end{aligned}$$

We know that $\displaystyle\int\sec\theta\ d\theta = \ln\left|\sec\theta+\tan\theta\right|+C$ and from the previous computation we have that $\displaystyle\int\sec^3\theta\ d\theta = \frac{1}{2}\sec\theta\tan\theta+\frac{1}{2}\ln\left|\sec\theta+\tan\theta\right|+C$. So that the integral is

$$\begin{aligned}
\int \frac{x^2}{\sqrt{x^2+9}}\,dx &= 9\int\sec^3\theta\ d\theta - 9\int\sec\theta\ d\theta = 9\left(\frac{1}{2}\sec\theta\tan\theta+\frac{1}{2}\ln\left|\sec\theta+\tan\theta\right|\right)-9\ln\left|\sec\theta+\tan\theta\right|+C\\
&= \frac{9}{2}\sec\theta\tan\theta+\frac{9}{2}\ln\left|\sec\theta+\tan\theta\right|-9\ln\left|\sec\theta+\tan\theta\right|+C = \frac{9}{2}\sec\theta\tan\theta-\frac{9}{2}\ln\left|\sec\theta+\tan\theta\right|+C
\end{aligned}$$

Now we need to reverse the substitution and write the result as an expression of x. Recall that $x = 3\tan\theta$. Then $\tan\theta = \dfrac{x}{3}$ and

$$\sec\theta = \sqrt{\tan^2\theta+1} = \sqrt{\left(\frac{x}{3}\right)^2+1} = \sqrt{\frac{1}{9}x^2+1} = \sqrt{\frac{1}{9}\left(x^2+9\right)} = \frac{1}{3}\sqrt{x^2+9}$$

and so

$$\begin{aligned}
\int \frac{x^2}{\sqrt{x^2+9}}\,dx &= \frac{9}{2}\sec\theta\tan\theta-\frac{9}{2}\ln\left|\sec\theta+\tan\theta\right|+C = \frac{9}{2}\left(\frac{1}{3}\sqrt{x^2+9}\right)\left(\frac{x}{3}\right)-\frac{9}{2}\ln\left|\left(\frac{1}{3}\sqrt{x^2+9}\right)+\frac{x}{3}\right|+C\\
&= \frac{1}{2}x\sqrt{x^2+9}-\frac{9}{2}\ln\left|\frac{x+\sqrt{x^2+9}}{3}\right|+C = \frac{1}{2}x\sqrt{x^2+9}-\frac{9}{2}\left(\ln\left|x+\sqrt{x^2+9}\right|-\ln 3\right)+C\\
&= \frac{1}{2}x\sqrt{x^2+9}-\frac{9}{2}\ln\left|x+\sqrt{x^2+9}\right|+\frac{9}{2}\ln 3+C = \boxed{\frac{1}{2}x\sqrt{x^2+9}-\frac{9}{2}\ln\left|x+\sqrt{x^2+9}\right|+C}
\end{aligned}$$

Example 7: $\displaystyle\int \frac{x^6}{\sqrt{1-x^{14}}}\,dx$

Solution: Let $u = x^7$. Then $du = 7x^6dx$ and so $dx = \dfrac{du}{7x^6}$. Then the integral becomes

$$\int \frac{x^6}{\sqrt{1-u^2}}\,\frac{du}{7x^6} = \frac{1}{7}\int\frac{1}{\sqrt{1-u^2}}\,du$$

We can either recognize that this is the derivative of $\sin^{-1}u$:

$$\frac{1}{7}\int\frac{1}{\sqrt{1-u^2}}\,du = \frac{1}{7}\sin^{-1}u+C = \frac{1}{7}\sin^{-1}\left(x^7\right)+C$$

If we do not recognize the derivative, then we can use trigonometric substitution $\theta = \sin^{-1} u$. Then $u = \sin\theta$ and so $du = \cos\theta d\theta$ and θ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$$\frac{1}{7}\int \frac{1}{\sqrt{1-u^2}}\ du = \frac{1}{7}\int \frac{1}{\sqrt{1-\sin^2\theta}}\ \cos\theta d\theta = \frac{1}{7}\int \frac{1}{\sqrt{\cos^2\theta}}\ \cos\theta d\theta = \frac{1}{7}\int \frac{1}{|\cos\theta|}\ \cos\theta d\theta$$

Since θ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\cos\theta$ is non-negative and so $|\cos\theta| = \cos\theta$ and so

$$\frac{1}{7}\int \frac{1}{|\cos\theta|}\ \cos\theta d\theta = \frac{1}{7}\int \frac{1}{\cos\theta}\ \cos\theta d\theta = \frac{1}{7}\int 1 d\theta = \frac{1}{7}\theta + C = \frac{1}{7}\sin^{-1}\left(x^7\right) + C$$

So the answer is $\boxed{\frac{1}{7}\sin^{-1}\left(x^7\right) + C}$.

Example 8: $\displaystyle\int_0^1 \frac{\tan^{-1} x}{x^2+1}\ dx$

Solution: Let $u = \tan^{-1} x$. Then $du = \frac{1}{1+x^2}dx$. For the limits of the integral, when $x = 0$, then $u = \tan^{-1}(0) = 0$ and when $x = 1$, $u = \tan^{-1}(1) = \frac{\pi}{4}$. So our integral becomes

$$\int_0^1 \frac{\tan^{-1} x}{x^2+1}\ dx = \int_0^{\pi/4} u\ du = \left.\frac{u^2}{2}\right|_0^{\pi/4} = \frac{1}{2}\left(\left(\frac{\pi}{4}\right)^2 - 0^2\right) = \frac{1}{2}\cdot\frac{\pi^2}{16} = \boxed{\frac{\pi^2}{32}}$$

Sample Problems

1. $\int\limits_{1}^{\infty} \frac{1}{x^4}\, dx$
2. $\int\limits_{1}^{\infty} \frac{1}{x}\, dx$
3. $\int\limits_{10}^{\infty} \frac{1}{x \ln x}\, dx$
4. $\int\limits_{0}^{\infty} e^{-5x}\, dx$
5. $\int\limits_{0}^{\infty} x e^{-x^2}\, dx$
6. $\int\limits_{0}^{\infty} x e^{-5x}\, dx$
7. $\int\limits_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$
8. $\int\limits_{0}^{1} \frac{1}{x} dx$
9. $\int\limits_{0}^{1} \frac{1}{\sqrt{x}} dx$
10. $\int\limits_{0}^{1} \ln x\, dx$
11. $\int\limits_{-1}^{4} \frac{x}{x^2-9} dx$
12. $\int\limits_{0}^{25} \frac{1}{\sqrt{x}} dx$
13. $\int\limits_{1}^{2} \frac{1}{\sqrt{2-x}} dx$
14. $\int\limits_{0}^{5} \frac{1}{\sqrt[3]{2-x}} dx$

Practice Problems

1. $\int\limits_{1}^{\infty} \frac{1}{x}\, dx$
2. $\int\limits_{1}^{\infty} \frac{1}{x^2}\, dx$
3. $\int\limits_{2}^{\infty} e^{-5x}\, dx$
4. $\int\limits_{0}^{1} \frac{1}{x}\, dx$
5. $\int\limits_{0}^{1} \frac{1}{x^2}\, dx$
6. $\int\limits_{0}^{1} \frac{1}{\sqrt[3]{x}}\, dx$
7. $\int\limits_{5}^{\infty} \frac{1}{x \ln x}\, dx$
8. $\int\limits_{0}^{1} \frac{1}{x \ln x}\, dx$
9. $\int\limits_{0}^{\infty} x e^{-2x}\, dx$
10. $\int\limits_{1}^{2} \frac{x^2}{x^3-8}\, dx$
11. $\int\limits_{1}^{\infty} \frac{1}{x^2+3}\, dx$
12. $\int\limits_{0}^{\pi/2} \tan x\, dx$
13. $\int\limits_{0}^{1} \frac{1}{\sqrt{1-x^2}}\, dx$
14. $\int\limits_{0}^{1} \frac{x}{\sqrt{1-x^2}}\, dx$
15. $\int\limits_{1}^{2} \frac{1}{x\sqrt{x^2-1}}\, dx$
16. $\int\limits_{2}^{\infty} \frac{1}{x\sqrt{x^2-1}}\, dx$
17. $\int\limits_{1}^{2} \frac{1}{(x-1)(x^2+1)}\, dx$
18. $\int\limits_{2}^{\infty} \frac{1}{(x-1)(x^2+1)}\, dx$
19. $\int\limits_{0}^{1} \frac{1}{\sqrt[3]{x}}\, dx$

Answers - Sample Problems

1.) $\frac{1}{3}$ 2.) ∞ 3.) ∞ 4.) $\frac{1}{5}$ 5.) $\frac{1}{2}$ 6.) $\frac{1}{25}$ 7.) π 8.) ∞ 9.) 2 10.) -1

11.) undefined 12.) 10 13.) 2 14.) $-\frac{3}{2}\left(\sqrt[3]{9}-\sqrt[3]{4}\right)$

Answers - Practice Problems

1.) ∞ 2.) 1 3.) $\frac{1}{5e^{10}}$ 4.) ∞ 5.) ∞ 6.) $\frac{3}{2}$ 7.) ∞ 8.) $-\infty$ 9.) $\frac{1}{4}$ 10.) $-\infty$

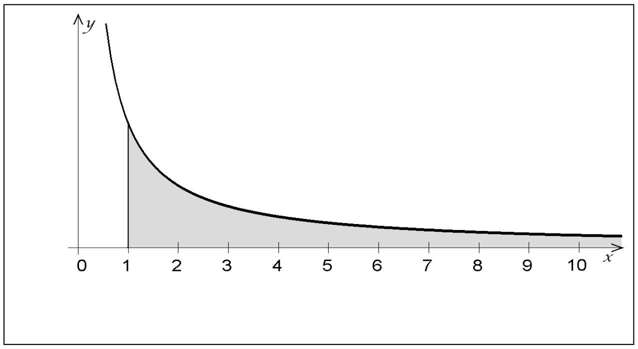
11.) $\frac{\pi\sqrt{3}}{9}$ 12.) ∞ 13.) $\frac{\pi}{2}$ 14.) 1 15.) $\frac{\pi}{3}$ 16.) $\frac{\pi}{6}$ 17.) ∞

18.) $\frac{1}{4}\ln 5-\frac{\pi}{4}+\frac{1}{2}\tan^{-1}2\approx 0.170\,535\,7$ 19.) $\frac{3}{2}$

Sample Problems - Solutions

Infinite Limits of Integration

There are two types of improper integrals. The ones with infinite limits of integration are easy to recognize, we are asked about the area of a region that is infinitely long. For example, $\int_1^\infty \frac{1}{x}\,dx$ and $\int_1^\infty \frac{1}{x^4}\,dx$ are such integrals. Let N be a very large positive number. The definite integral $\int_1^N \frac{1}{x^4}\,dx$ is defined for all positive N. So what we do is we let N approach infinity and determine what the values of the definite integrals are doing. If they approach a finite number, we define that to be the area under the graph. If the limit of the definite integrals is infinte, we say that the area under the graph is infinite, and the integral diverges.



1. $\int_1^\infty \frac{1}{x^4}dx = \frac{1}{3}$

Solution: We compute the limit of the definite integrals as the upper limit approaches infinity.

$$\int_1^\infty \frac{1}{x^4}dx = \lim_{N\to\infty}\int_1^N x^{-4}dx = \lim_{N\to\infty}\left.\frac{x^{-3}}{-3}\right|_1^N = \lim_{N\to\infty}\left(\frac{N^{-3}}{-3}-\frac{1^{-3}}{-3}\right) = \lim_{N\to\infty}\left(\underset{\overset{\downarrow}{0}}{\frac{-1}{3N^3}}-\left(\frac{-1}{3}\right)\right) = \frac{1}{3}$$

2. $\int_1^\infty \frac{1}{x}dx = \infty$

Solution: We compute the limit of the definite integrals as the upper limit approaches infinity.

$$\int_1^\infty \frac{1}{x}dx = \lim_{N\to\infty}\int_1^N \frac{1}{x}dx = \lim_{N\to\infty}\ln|x|\Big|_1^N = \lim_{N\to\infty}\left(\underset{\overset{\downarrow}{\infty}}{\ln N}-\ln 1\right) = \infty$$

This improper integral diverges.

3. $\displaystyle\int_{10}^{\infty} \frac{1}{x\ln x}dx = \infty$

Solution: We first compute the indefinite integral by substitution. Let $u = \ln x$. Then $du = \frac{1}{x}dx$.

$$\int \frac{1}{x\ln x}dx = \int \frac{1}{\ln x}\left(\frac{1}{x}dx\right) = \int \frac{1}{u}du = \ln|u| + C = \ln|\ln x| + C$$

Now we are ready to evaluate the improper integral.

$$\int_{10}^{\infty} \frac{1}{x\ln x}dx = \lim_{N\to\infty}\int_{10}^{N} \frac{1}{x\ln x}dx = \lim_{N\to\infty} \ln\ln|x|\Big|_{10}^{N} = \lim_{N\to\infty}\left(\ln\big(\underset{\infty}{\underset{\downarrow}{\ln N}}\big) - \ln(\ln 10)\right) = \infty$$

This improper integral diverges.

4. $\displaystyle\int_{0}^{\infty} e^{-5x}dx = \frac{1}{5}$

Solution:

$$\int_{0}^{\infty} e^{-5x}dx = \lim_{N\to\infty}\int_{0}^{N} e^{-5x}dx = \lim_{N\to\infty}\frac{e^{-5x}}{-5}\Big|_{0}^{N} = \lim_{N\to\infty}\left(\frac{e^{-5N}}{-5} - \frac{e^{-5(0)}}{-5}\right) = \lim_{N\to\infty}\left(\underset{0}{\underset{\downarrow}{\frac{-1}{-5e^{5N}}}} - \frac{1}{-5}\right) = \frac{1}{5}$$

5. $\displaystyle\int_{0}^{\infty} xe^{-x^2}dx = \frac{1}{2}$

Solution: We first compute the indefinite integral, by substitution. Let $u = -x^2$. Then $du = -2xdx$.

$$\int xe^{-x^2}dx = \int e^{-x^2}(xdx) = \int e^u\left(-\frac{1}{2}du\right) = -\frac{1}{2}\int e^u du = -\frac{1}{2}e^u + C = -\frac{1}{2}e^{-x^2} + C$$

Now we are ready to evaluate the improper integral.

$$\int_{0}^{\infty} xe^{-x^2}dx = \lim_{N\to\infty}\int_{0}^{N} xe^{-x^2}dx = \lim_{N\to\infty} -\frac{1}{2}e^{-x^2}\Big|_{0}^{N} = -\frac{1}{2}\lim_{N\to\infty}\frac{1}{e^{x^2}}\Big|_{0}^{N} = -\frac{1}{2}\lim_{N\to\infty}\left(\underset{0}{\underset{\downarrow}{\frac{1}{e^{N^2}}}} - \frac{1}{e^{0^2}}\right) = \frac{1}{2}$$

6. $\displaystyle\int_{0}^{\infty} xe^{-5x}dx = \frac{1}{25}$

Solution: We first find the antiderivative by integrating by parts. Let $u = x$ and $dv = e^{-5x}dx$. Then $du = dx$ and

$$v = \int dv = \int e^{-5x}dx = -\frac{1}{5}e^{-5x} + C$$

So we have

$u = x$	$v = -\frac{1}{5}e^{-5x}$
$du = dx$	$dv = e^{-5x}dx$

$$\begin{aligned}
\int u\ dv &= uv - \int v\ du \quad \text{becomes} \\
\int x\ e^{-5x}dx &= -\frac{1}{5}xe^{-5x} - \int -\frac{1}{5}e^{-5x}\ dx = -\frac{1}{5}xe^{-5x} + \frac{1}{5}\int e^{-5x}\ dx = -\frac{1}{5}xe^{-5x} + \frac{1}{5}\left(-\frac{1}{5}e^{-5x}\right) + C \\
&= -\frac{1}{5}xe^{-5x} - \frac{1}{25}e^{-5x} + C
\end{aligned}$$

Now for the improper integral:

$$\begin{aligned}
\int_0^\infty x\ e^{-5x}dx &= \lim_{N\to\infty}\int_0^N x\ e^{-5x}dx = \lim_{N\to\infty}\left(-\frac{1}{5}xe^{-5x} - \frac{1}{25}e^{-5x}\right)\Big|_0^N \\
&= \lim_{N\to\infty}\left(\left(-\frac{1}{5}Ne^{-5N} - \frac{1}{25}e^{-5N}\right) - \left(-\frac{1}{5}(0)\,e^{-5(0)} - \frac{1}{25}e^{-5(0)}\right)\right) \\
&= \lim_{N\to\infty}\left(\left(-\frac{1}{5}\frac{N}{e^{5N}} - \frac{1}{25e^{5N}}\right) - \left(-\frac{1}{25}\right)\right) = \lim_{N\to\infty}\left(\frac{1}{25} - \frac{1}{5}\underset{\underset{?}{\downarrow}}{\frac{N}{e^{5N}}} - \underset{\underset{0}{\downarrow}}{\frac{1}{25e^{5N}}}\right)
\end{aligned}$$

The limit $\lim_{N\to\infty}\frac{N}{e^{5N}}$ is an $\frac{\infty}{\infty}$ type of an indeterminate. We apply L'Hôpital's rule:

$$\lim_{N\to\infty}\frac{N}{e^{5N}} = \lim_{N\to\infty}\frac{1}{5e^{5N}} = 0$$

Thus

$$\int_0^\infty x\ e^{-5x}dx = \lim_{N\to\infty}\left(\frac{1}{25} - \frac{1}{5}\underset{\underset{0}{\downarrow}}{\frac{N}{e^{5N}}} - \underset{\underset{0}{\downarrow}}{\frac{1}{25e^{5N}}}\right) = \frac{1}{25}$$

7. $\displaystyle\int_{-\infty}^{\infty}\frac{1}{1+x^2}\ dx = \pi$

Solution: Both limits of this integral are infinite. In case of such an integral, we separate it to a sum of two improper integrals

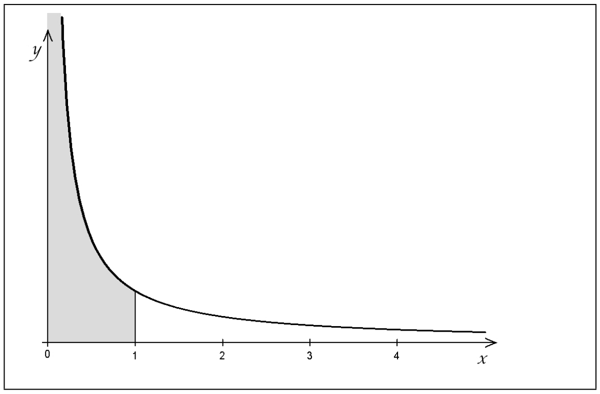
$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}\ dx = \int_{-\infty}^{0}\frac{1}{1+x^2}\ dx + \int_0^{\infty}\frac{1}{1+x^2}\ dx$$

Because $f(x) = \frac{1}{1+x^2}$ is an even function, the two integrals are the same:

$$\begin{aligned}
\int_{-\infty}^{\infty}\frac{1}{1+x^2}\ dx &= \int_{-\infty}^{0}\frac{1}{1+x^2}\ dx + \int_0^{\infty}\frac{1}{1+x^2}\ dx = 2\int_0^{\infty}\frac{1}{1+x^2}\ dx = 2\lim_{N\to\infty}\int_0^N\frac{1}{1+x^2}\ dx \\
&= 2\lim_{N\to\infty}\left(\tan^{-1}x\right)\Big|_0^N = 2\lim_{N\to\infty}\left(\tan^{-1}N - \tan^{-1}0\right) = 2\left(\frac{\pi}{2} - 0\right) = \pi
\end{aligned}$$

Integrands with Vertical Asymptotes

Some integrals are improper because they represent an infinitely tall region. These are not trickier to compute but more difficult to detect. For example, $\int_0^1 \frac{1}{x}dx$ appears to be a definite integral. However, there is a vertical asymptote at zero, making the integral improper.



8. $\int_0^1 \frac{1}{x}dx = \infty$

   Solution: We compute the limit of definite integrals as the lower limit approaches zero.

   $$\int_0^1 \frac{1}{x}dx = \lim_{h\to 0^+} \int_h^1 \frac{1}{x}dx = \lim_{h\to 0^+} \left(\ln|x|\right)\Big|_h^1 = \lim_{h\to 0^+} \left(\ln|1| - \ln|h|\right) = \lim_{h\to 0^+} \left(0 - (-\infty)\right) = \infty$$

   The area under the graph is infinite; this integral diverges.

9. $\int_0^1 \frac{1}{\sqrt{x}}dx = 2$

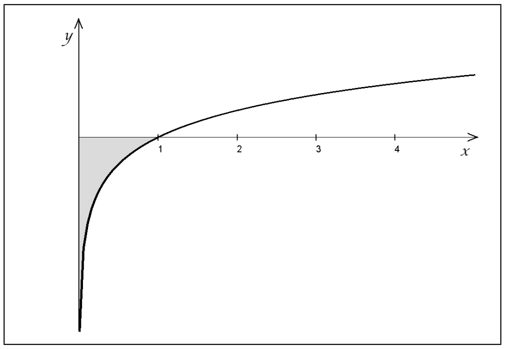
   Solution: We compute the limit of definite integrals as the lower limit approaches zero.

   $$\int_0^1 \frac{1}{\sqrt{x}}dx = \lim_{h\to 0^+} \int_h^1 x^{-1/2}dx = \lim_{h\to 0^+} \left(2\sqrt{x}\right)\Big|_h^1 = \lim_{h\to 0^+} \left(2\sqrt{1} - 2\sqrt{h}\right) = 2$$

   The area under the graph is 2.

10. $\displaystyle\int_0^1 \ln x \; dx = -1$

Solution: This is an improper integral because there is a vertical asymptote at zero.



We compute this integral by taking the limits of definite integrals, between a small positive number and 1.

$$\int_0^1 \ln x dx = \lim_{h \to 0^+} \int_h^1 \ln x dx$$

We compute the antiderivative of $\ln x$ by integration by parts. Let $u = \ln x$ and $dv = dx$. Then $du = \dfrac{1}{x} dx$ and $v = x$.

$$\begin{aligned} \int u \; dv &= uv - \int v \; du \text{ becomes} \\ \int \ln x \; dx &= x \ln x - \int x \left(\frac{1}{x} dx\right) = x \ln x - \int \; dx = x \ln x - x + C \end{aligned}$$

The improper integral is

$$\begin{aligned} \int_0^1 \ln x dx &= \lim_{h \to 0^+} \int_h^1 \ln x dx = \lim_{h \to 0^+} \left. (x \ln x - x) \right|_h^1 = \lim_{h \to 0^+} \left((1 \ln 1 - 1) - (h \ln h - h) \right) \\ &= \lim_{h \to 0^+} \left(-1 - \underset{?}{\underset{\downarrow}{h \ln h}} - \underset{0}{\underset{\downarrow}{h}} \right) \end{aligned}$$

As h approaches zero from the right, $\ln h$ approaches negative infinity. This means that $\displaystyle\lim_{h \to 0^+} h \ln h = \lim_{h \to 0^+} \frac{\ln h}{\frac{1}{h}}$ is an $\dfrac{\infty}{\infty}$ type of an indeterminate. We apply L'Hôpital's rule:

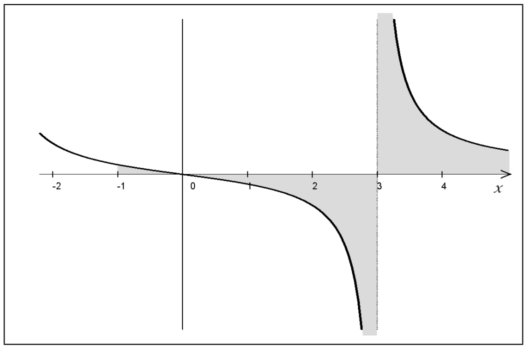
$$\lim_{h \to 0^+} \frac{\ln h}{\frac{1}{h}} = \lim_{h \to 0^+} \frac{\frac{1}{h}}{-\frac{1}{h^2}} = \lim_{h \to 0^+} \frac{1}{h} \left(\frac{-h^2}{1} \right) = \lim_{h \to 0^+} (-h) = 0$$

Thus the improper integral is

$$\int_0^1 \ln x dx = \lim_{h \to 0^+} \left(-1 - \underset{0}{\underset{\downarrow}{h \ln h}} - \underset{0}{\underset{\downarrow}{h}} \right) = -1$$

11. $\displaystyle\int_{-1}^{4} \frac{x}{x^2-9}\,dx = \text{undefined}$

Solution: This is an imporper integral because there is a vertical asymptote at $x = 3$.



We again separate this into two improper integrals, the area of the region to the left of 3 and to the right of 3.

$$\int_{-1}^{4} \frac{x}{x^2-9}dx = \int_{-1}^{3} \frac{x}{x^2-9}dx + \int_{3}^{4} \frac{x}{x^2-9}dx = \lim_{a\to 3^-}\int_{-1}^{a} \frac{x}{x^2-9}dx + \lim_{b\to 3^+}\int_{b}^{4} \frac{x}{x^2-9}dx$$

We first compute the indefinite integral by substitution. Let $u = x^2 - 9$. Then $du = 2xdx$.

$$\int \frac{x}{x^2-9}dx = \int \frac{1}{x^2-9}(xdx) = \int \frac{1}{u}\left(\frac{1}{2}du\right) = \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln\left|x^2-9\right| + C$$

Now we are ready to evaluate the improper integral.

$$\begin{aligned}\int_{-1}^{3} \frac{x}{x^2-9}dx &= \lim_{a\to 3^-}\int_{-1}^{a} \frac{x}{x^2-9}dx = \lim_{a\to 3^-}\frac{1}{2}\ln\left|x^2-9\right|\Big|_{-1}^{a} = \frac{1}{2}\lim_{a\to 3^-}\ln\left|x^2-9\right|\Big|_{-1}^{a} \\ &= \frac{1}{2}\lim_{a\to 3^-}\left(\ln\left|a^2-9\right| - \ln\left|(-1)^2-9\right|\right) = \frac{1}{2}\lim_{a\to 3^-}\left(\ln\left|a^2-9\right| - \ln 8\right) = -\infty\end{aligned}$$

The other part:

$$\begin{aligned}\int_{3}^{4} \frac{x}{x^2-9}dx &= \lim_{b\to 3^+}\int_{b}^{4} \frac{x}{x^2-9}dx = \lim_{b\to 3^+}\frac{1}{2}\ln\left|x^2-9\right|\Big|_{b}^{4} = \lim_{b\to 3^+}\left(\frac{1}{2}\ln\left|4^2-9\right| - \frac{1}{2}\ln\left|b^2-9\right|\right) \\ &= \lim_{b\to 3^+}\left(\frac{1}{2}\ln 7 - \frac{1}{2}\ln\left|b^2-9\right|\right) = \infty\end{aligned}$$

When we combine two improper integrals, finite sums are allowed to be added, such as in problem #7. However, the sum $\infty + (-\infty)$ is an indeterminate and in this case, it is not defined. This integral is undefined. We also say that the area under the graph is undefined.

12. $\displaystyle\int_{0}^{25} \frac{1}{\sqrt{x}}dx = 10$

Solution:

$$\int_{0}^{25} \frac{1}{\sqrt{x}}dx = \lim_{a\to 0^+}\int_{a}^{25} x^{-1/2}dx = \lim_{a\to 0^+} 2\sqrt{x}\Big|_{a}^{25} = 2\lim_{a\to 0^+}\sqrt{x}\Big|_{a}^{25} = 2\lim_{a\to 0^+}\left(\sqrt{25} - \sqrt{a}\right) = 10$$

13. $\displaystyle\int_1^2 \frac{1}{\sqrt{2-x}}dx = 2$

Solution: We first compute the indefinite integral, by substitution. Let $u = 2-x$. Then $du = -dx$ and so $dx = -du$.

$$\int \frac{1}{\sqrt{2-x}}dx = \int \frac{1}{\sqrt{u}}\left(-du\right) = -\int u^{-1/2}du = -2u^{1/2}+C = -2\sqrt{2-x}+C$$

Now we are ready to evaluate the improper integral.

$$\begin{aligned}\int_1^2 \frac{1}{\sqrt{2-x}}dx &= \lim_{a\to 2^-}\int_1^a \frac{1}{\sqrt{2-x}}dx = \lim_{a\to 2^-} -2\sqrt{2-x}\Big|_1^a = -2\lim_{a\to 2^-}\sqrt{2-x}\Big|_1^a \\ &= -2\lim_{a\to 2^-}\left(\sqrt{2-a}-\sqrt{2-1}\right) = -2\lim_{a\to 2^-}\left(\sqrt{2-2}-\sqrt{1}\right) = -2\left(-1\right) = 2\end{aligned}$$

14. $\displaystyle\int_0^5 \frac{1}{\sqrt[3]{2-x}}dx = -\frac{3}{2}\left(\sqrt[3]{9}-\sqrt[3]{4}\right)$

Solution: We first compute the indefinite integral, by substitution. Let $u = 2-x$. Then $du = -dx$ and so $dx = -du$

$$\int \frac{1}{\sqrt[3]{2-x}} = \int \frac{1}{\sqrt[3]{u}}\left(-du\right) = -\int u^{-1/3}du = -\frac{3}{2}u^{2/3}+C = -\frac{3}{2}\left(2-x\right)^{2/3}+C$$

Now we are ready to evaluate the improper integral.

$$\begin{aligned}\int_0^5 \frac{1}{\sqrt[3]{2-x}}dx &= \int_0^2 \frac{1}{\sqrt[3]{2-x}}dx + \int_2^5 \frac{1}{\sqrt[3]{2-x}}dx = \lim_{a\to 2^-}\int_0^a \frac{1}{\sqrt[3]{2-x}}dx + \lim_{b\to 2^+}\int_b^5 \frac{1}{\sqrt[3]{2-x}}dx \\ &= \lim_{a\to 2^-} -\frac{3}{2}\left(2-x\right)^{2/3}\Big|_0^a + \lim_{b\to 2^+} -\frac{3}{2}\left(2-x\right)^{2/3}\Big|_b^5 = -\frac{3}{2}\left(\lim_{a\to 2^-}\left(2-x\right)^{2/3}\Big|_0^a + \lim_{b\to 2^+}\left(2-x\right)^{2/3}\Big|_b^5\right) \\ &= -\frac{3}{2}\left(\lim_{a\to 2^-}\left(\left(2-a\right)^{2/3}-\left(2-0\right)^{2/3}\right) + \lim_{b\to 2^+}\left(\left(2-5\right)^{2/3}-\left(2-b\right)^{2/3}\right)\right) \\ &= -\frac{3}{2}\left(-2^{2/3}+\left(-3\right)^{2/3}\right) = -\frac{3}{2}\left(-\sqrt[3]{4}+\sqrt[3]{9}\right) = -\frac{3}{2}\left(\sqrt[3]{9}-\sqrt[3]{4}\right)\end{aligned}$$

5–14 ■ Solve the differential equation.

5. $y' + y = 1$

6. $y' - y = e^x$

7. $y' = x - y$

8. $4x^3y + x^4y' = \sin^3 x$

9. $xy' + y = \sqrt{x}$

10. $y' + y = \sin(e^x)$

11. $\sin x \dfrac{dy}{dx} + (\cos x)y = \sin(x^2)$

12. $x \dfrac{dy}{dx} - 4y = x^4e^x$

13. $(1 + t) \dfrac{du}{dt} + u = 1 + t, \quad t > 0$

14. $t \ln t \dfrac{dr}{dt} + r = te^t$

15–20 ■ Solve the initial-value problem.

15. $x^2y' + 2xy = \ln x, \quad y(1) = 2$

16. $t^3 \dfrac{dy}{dt} + 3t^2y = \cos t, \quad y(\pi) = 0$

17. $t \dfrac{du}{dt} = t^2 + 3u, \quad t > 0, \quad u(2) = 4$

18. $2xy' + y = 6x, \quad x > 0, \quad y(4) = 20$

19. $xy' = y + x^2 \sin x, \quad y(\pi) = 0$

20. $(x^2 + 1) \dfrac{dy}{dx} + 3x(y - 1) = 0, \quad y(0) = 2$