HART

1951

A SECOND COURSE IN ALGEBRA

SECOND EDITION ENLARGED

WITH ANSWERS

HEATH

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SECOND EDITION ENLARGED

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1. If $y = f(x)$, what is meant by the symbol:

(*a*) Dx? (*b*) Dy? (*c*) $\dfrac{Dy}{Dx}$? (*d*) $\dfrac{dy}{dx}$?

2. If (x_1, y_1) is a point on the graph of $y = f(x)$, what is the geometrical interpretation of: (*a*) $\dfrac{Dy}{Dx}$? (*b*) $\dfrac{dy}{dx}$?

3. What is the slope of a tangent to a graph when the tangent is parallel to the X-axis?

4. If $y = a + bx + cx^2 + dx^3 + \cdots$, what is $\dfrac{dy}{dx}$?

5. What is $\dfrac{dy}{dx}$ when $y = Ax^4 + Bx^3 + Cx^2 + Dx$?

Find the derivative when:

6. $y = 2x$ **7.** $y = 3x + 5$ **8.** $y = x^2 + 2x -$

9. $y = x^2$ **10.** $y = 2x^2 - 3x$ **11.** $y = x^2 - 3x +$

12. $y = x^3$ **13.** $y = 2x^3 + 3x$ **14.** $y = x^3 - 2x -$

15. $y = 3x^3 - 2x^2 + x - 4$ **16.** $y = 4x^3 - 3x^2 + x -$

17. Find the value of the derivatives in Examples 6 to 16 when x is: (*a*) $+2$ (*b*) $-\frac{1}{2}$

18. What is the slope of the tangent to $y = x^2 - 5x + 2$ at $x = 1$?

Note. This is also called the slope of the curve at $x = 1$.

19. Is there a point where the tangent to $y = 2 - x + x^2$ is parallel to the X-axis? If there is, find both coordinates of the point.

20. (*a*) Is there a point where the tangent to $y = 2x^2 + x -$ is parallel to the X-axis? If there is, find both coordinates of the point.

(*b*) Also find the equation of the tangent to the graph at that point.

21. Write the equation of the tangent to $y = 5 - x^2$ at the point where $x = 1$.

22. (*a*) If the graph of $y = x^3 + x^2 - x + 3$ has any points where its slope is zero, find their coordinates.

(*b*) Find the equations of the tangents to the graph at any such points.

In Chapter 3, you learned how to combine two or more fractions into a single fraction.

Thus:

$$\frac{2}{x-1} + \frac{3x}{x+1} = \frac{2x + 2 + 3x^2 - 3x}{x^2 - 1}$$

$$= \frac{3x^2 - x + 2}{x^2 - 1}$$

In some parts of mathematics it is necessary to *do the reverse problem;* namely, to *find the fractions with prime denominators that equal a given fraction when they are combined.* (*See the last page to decompose back to the original state.*)

This is called finding the **partial fractions** whose algebraic sum equals a given fraction.

The following theorem about polynomials must be proved first.

Theorem. *If two polynomials $A(x)$ and $B(x)$ of degree n are equal for more than n values of x, then the coefficients of like powers of x in the two polynomials must be equal.*

PROOF.

1. Let $A(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$
   and $B(x) = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n$
   and let $A(x) = B(x)$ for more than n values of x.
2. From 1, by subtraction:
   $(a_0 - b_0) + (a_1 - b_1)x + (a_2 - b_2)x^2 + \cdots + (a_n - b_n)x^n = 0$
   for more than n values of x.
3. But if every coefficient in Step 2, including $(a_0 - b_0)$, $\neq 0$, then we have an equation of degree n with more than n roots and this is impossible.
4. $\therefore$ $a_0 - b_0 = 0;\ a_1 - b_1 = 0;\ a_2 - b_2 = 0;\ \cdots\ a_n - b_n = 0,$
   or $a_0 = b_0,\ a_1 = b_1,\ a_2 = b_2,\ a_n = b_n.$

We are prepared next to take up the study of the principal part of Topic F.

There are several cases or types of such problems and they will be considered in turn.

Case I. When there are no equal factors in the denominator.

Example 1. Separate $\dfrac{19x+1}{(3x-1)(5x+2)}$ into partial fractions.

Solution.

1. Let $$\frac{19x+1}{(3x-1)(5x+2)} = \frac{A}{3x-1} + \frac{B}{5x+2}, \qquad (1)$$
where A and B are numbers independent of x.

2. $\therefore \quad 19x + 1 = A(5x+2) + B(3x-1)$
or $\quad 19x + 1 = (5A + 3B)x + 2A - B \qquad (2)$

3. Equation (1) must be true for every value of x except $x = \frac{1}{3}$ and $-\frac{2}{5}$.

4. Therefore the coefficients of like powers of x in the two members of equation (2) are equal. That is, $5A + 3B = 19$, and $2A - B = 1$.

5. Solving these equations, $A = 2$ and $B = 3$.

6. Substituting in (1), $\dfrac{19x+1}{(3x-1)(5x+2)} = \dfrac{2}{3x-1} + \dfrac{3}{5x+2}$

Example 2. Separate $\dfrac{x+4}{2x-x^2-x^3}$ into partial fractions.

Solution.

1. $$\frac{x+4}{2x-x^2-x^3} = \frac{x+4}{x(2-x-x^2)} = \frac{x+4}{x(2+x)(1-x)}$$

2. Assume $$\frac{x+4}{2x-x^2-x^3} = \frac{A}{x} + \frac{B}{1-x} + \frac{C}{2+x}. \qquad (3)$$

3. $\therefore \quad x + 4 = A(1-x)(2+x) + Bx(2+x) + Cx(1-x)$

4. $\therefore \quad x + 4 = A(2 - x - x^2) + 2Bx + Bx^2 + Cx - Cx^2$
$\quad = 2A + x(-A + 2B + C) + x^2(-A + B - C) \qquad (4)$

5. Since equation (4) is true for more than two values of x:

$\therefore \quad 2A = 4, \quad$ or $\quad A = 2$

$-A + 2B + C = 1, \quad$ and $\quad -A + B - C = 0$

$\therefore \quad 2B + C - 2 = 1, \quad$ or $\quad 2B + C = 3$

and $\quad B - C - 2 = 0, \quad$ or $\quad B - C = 2$

6. Solving the system $\begin{cases} 2B + C = 3 \\ B - C = 2 \end{cases}$ $B = \frac{5}{3}$, and $C = -$

7. $$\therefore \quad \frac{x+4}{2x-x^2-x^3} = \frac{2}{x} + \frac{5}{3(1-x)} - \frac{1}{3(2+x)}$$

Separate each of the following fractions into partial fractions:

1. $\dfrac{6}{x^2 - 3x - 10}$

2. $\dfrac{2x-5}{x^2 - 5x - 14}$

3. $\dfrac{3x+2}{x^3 - 16x}$

4. $\dfrac{8x}{x^2 - 4}$

5. $\dfrac{2a+15}{9 - a^2}$

6. $\dfrac{3y^2 + 5y - 2}{y(y^2 - 1)}$

7. $\dfrac{13 - 4m}{10m^2 + 11m - 6}$

8. $\dfrac{29p - 12}{10p^2 - 23p + 12}$

9. $\dfrac{38c + 5}{6c^2 + 5c - 6}$

10. $\dfrac{ax - 19a^2}{x^2 + 4ax - 5a^2}$

11. $\dfrac{46t - 5x}{8t^2 - 18xt - 5x^2}$

12. $\dfrac{x^2 + 10x - 7}{(2x-1)(12x^2 - x - 6)}$

13. $\dfrac{27x - 13x^2 + 18}{(x^2 - 2x)(x^2 - 9)}$

14. $\dfrac{4x^2 + 9x + 12}{(2x-1)(x^2 + 5x + 6)}$

15. $\dfrac{3x - 20}{x^2 + 3x - 10}$

16. $\dfrac{5x^2 + 13x}{(x+1)(2 - x - x^2)}$

Case II. When all the factors of the denominator are equal and of degree one.

Let it be required to separate $\dfrac{x^2 - 11x + 26}{(x-3)^3}$ into partial fractions.

Substituting $y + 3$ for x, the fraction becomes:

$$\frac{(y+3)^2 - 11(y+3) + 26}{y^3} = \frac{y^2 - 5y + 2}{y^3} = \frac{1}{y} - \frac{5}{y^2} + \frac{2}{y^3}$$

Replacing y by $x - 3$, the result takes the form:

$$\frac{1}{x-3} - \frac{5}{(x-3)^2} + \frac{2}{(x-3)^3}$$

This shows that the given fraction can be expressed as the sum of three partial fractions, whose numerators are independent of x, and whose denominators are the powers of $x - 3$ beginning with the first and ending with the third. The number of partial fractions is the same as the number of equal factors in the denominator of the given fraction.

EXAMPLE. Separate $\dfrac{6x+5}{(3x+5)^2}$ into partial fractions.

SOLUTION.

1. Assume the given fraction equal to the sum of *two* partial fractions, whose denominators are the powers of $3x + 5$ beginning with the first and ending with the *second*.

Thus:
$$\frac{6x+5}{(3x+5)^2} = \frac{A}{3x+5} + \frac{B}{(3x+5)^2}$$

2. $$\therefore\ 6x + 5 = A(3x+5) + B$$
or
$$6x + 5 = 3Ax + 5A + B$$

3. Equating the coefficients of like powers of x,
$$3A = 6$$
$$5A + B = 5$$

4. $$\therefore\ A = 2 \text{ and } B = -5$$

5. Whence $$\frac{6x+5}{(3x+5)^2} = \frac{2}{3x+5} - \frac{5}{(3x+5)^2}$$

The solution can be checked, of course, by combining the fractions of the right member of the equation of Step 5.

Separate each of the following fractions into partial fractions:

1. $\dfrac{3x-2}{(x-2)^2}$
2. $\dfrac{7m-5}{(m+3)^2}$
3. $\dfrac{2x^2+5}{(x-1)^3}$
4. $\dfrac{6x^2-x+2}{(x-3)^3}$
5. $\dfrac{2y-9}{(3y-2)^2}$
6. $\dfrac{14a-30}{4a^2-12a+9}$
7. $\dfrac{3x^3-2x+1}{(x-2)^4}$
8. $\dfrac{z^2+4z-1}{(z+5)^3}$
9. $\dfrac{6a-11}{9a^2-6a+1}$
10. $\dfrac{10m^2+3m-1}{(5m+2)^3}$
11. $\dfrac{c^3-3c^2-c}{(c-1)^4}$
12. $\dfrac{x^3+4x^2+7x+2}{(x+2)^4}$
13. $\dfrac{18x^3-21x^2+4x}{(3x-2)^4}$
14. $\dfrac{8x^3+4x^2-5}{(2x-1)^4}$
15. $\dfrac{-3x^2+7x-6}{(x-1)^3}$
16. $\dfrac{a^2+a}{(a+2)^3}$

Case III. When some of the factors of the denominator are equal.

EXAMPLE. Separate $\dfrac{x^2-4x+3}{x(x+1)^2}$ into partial fractions.

SOLUTION.

1. Case III is a combination of Cases I and II.
2. Assume $$\frac{x^2-4x+3}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$
3. Clearing of fractions, $$x^2 - 4x + 3 = A(x+1)^2 + Bx(x+1) + Cx = (A+B)x^2 + (2A+B+C)x + A$$
4. Equating the coefficients of like powers of x,
$$A + B = 1$$
$$2A + B + C = -4$$
$$A = 3$$
5. Solving these equations, $A = 3$, $B = -2$, and $C = -8$
6. Whence $$\frac{x^2-4x+3}{x(x+1)^2} = \frac{3}{x} - \frac{2}{x+1} - \frac{8}{(x+1)^2}$$

Note. A fraction of the form $\dfrac{X}{(x+a)(x+b)\cdots(x+m)^r\cdots}$ should be assumed equal to

$$\frac{A}{x+a} + \frac{B}{x+b} + \cdots + \frac{E}{x+m} + \frac{F}{(x+m)^2} + \cdots + \frac{K}{(x+m)^r} + \cdots$$

Factors like $x + a$ and $x + b$ have single partial fractions corresponding to them as in Case I; and repeated factors like $(x+m)^r$ have r partial fractions corresponding to them as in Case II.

Separate the following into partial fractions:

1. $\dfrac{3x^2-x+27}{x(x+3)^2}$
2. $\dfrac{3x^3+7x^2+24x-16}{x^3(x-4)}$
3. $\dfrac{14x^2-53x-4}{(3x+2)(2x-3)^2}$
4. $\dfrac{4x^3-x^2-7x-4}{x^2(x+1)^2}$
5. $\dfrac{-4x^3+29x^2-36x-9}{x(x-1)(x-3)^2}$
6. $\dfrac{7-13x-4x^2}{(8x^2-2x-3)(2x+1)}$
7. $\dfrac{3x^2+3x+4}{(2x-1)(3x+1)^2}$
8. $\dfrac{4x^3+6x^2+10x+3}{(6x^2-x-1)(2x+1)^2}$

We shall next assume that *the degree of the numerator is equal to or greater than that of the denominator.*

Divide the numerator by the denominator until a remainder is obtained which is of lower degree than the denominator.

EXAMPLE. Separate $\dfrac{x^3 - 3x^2 - 1}{x^2 - x}$ into an integral expression and partial fractions.

SOLUTION.

1. Dividing $x^3 - 3x^2 - 1$ by $x^2 - x$, the quotient is $x - 2$, and the remainder $-2x - 1$; then:

$$\frac{x^3 - 3x^2 - 1}{x^2 - x} = x - 2 + \frac{-2x - 1}{x^2 - x} \qquad (1)$$

2. Now separate $\dfrac{-2x - 1}{x^2 - x}$ into partial fractions by the method of Case I; the result is $\dfrac{1}{x} - \dfrac{3}{x - 1}$.

3. Substituting in (1), $\dfrac{x^3 - 3x^2 - 1}{x^2 - x} = x - 2 + \dfrac{1}{x} - \dfrac{3}{x - 1}$.

Separate each of the following fractions into an integral expression and two or more partial fractions:

1. $\dfrac{m^2 + 1}{m^2 - 1}$
2. $\dfrac{2r^2}{r^2 - 9}$
3. $\dfrac{2x^2 - 9}{9x - 9 - 2x^2}$
4. $\dfrac{3x^2 - x + 2}{6x^2 - 6}$
5. $\dfrac{r^2}{r^2 - 4}$
6. $\dfrac{12y^3 - 17y^2 + 7}{3y^2 - 5y - 2}$
7. $\dfrac{a^2 + 2a + 9}{6a^2 + 24a + 18}$
8. $\dfrac{10x^2 - 35x}{12x^2 + 18x - 12}$
9. $\dfrac{x^2 + 17x + 1}{x^2 + x - 12}$
10. $\dfrac{4a^2 - 4a + 1}{a^2 - 1}$
11. $\dfrac{8a^2 - 5a + 2}{4a^2 - 1}$
12. $\dfrac{3c^2 + 2c - 1}{c^2 - 4c + 4}$
13. $\dfrac{3x^3 - 2x^2 + 1}{x^2 - x - 2}$
14. $\dfrac{2x^4 - 5}{x^2 - 2x + 1}$

We shall consider, next, a fraction whose denominator has factors partly of the first and partly of the second degree, or all of the second degree, and whose numerator is of lower degree than the denominator. For a partial fraction having a denominator of the second degree, the numerator has the form $Ax + B$.

EXAMPLE. Separate $\dfrac{1}{x^3 + 1}$ into partial fractions.

SOLUTION.

1. The factors of the denominator are $x + 1$ and $x^2 - x + 1$.

2. Assume $\dfrac{1}{x^3 + 1} = \dfrac{A}{x + 1} + \dfrac{Bx + C}{x^2 - x + 1}$ (1)

3. $\therefore \quad 1 = A(x^2 - x + 1) + (Bx + C)(x + 1)$

or $\quad 1 = (A + B)x^2 + (-A + B + C)x + A + C$.

4. Equating the coefficients of like powers of x,

$$A + B = 0$$
$$-A + B + C = 0$$
$$A + C = 1$$

5. Solving these equations, $A = \frac{1}{3}$, $B = -\frac{1}{3}$, and $C = \frac{2}{3}$.

6. Substituting in (1), $\dfrac{1}{x^3 + 1} = \dfrac{1}{3(x + 1)} - \dfrac{x - 2}{3(x^2 - x + 1)}$.

Separate each of the following into partial fractions:

1. $\dfrac{5x^2 + 1}{x^3 + 1}$
2. $\dfrac{x^2 + 16x - 12}{(3x + 1)(x^2 - x + 3)}$
3. $\dfrac{2x^2 + 11x - 7}{(2x - 5)(x^2 + 2)}$
4. $\dfrac{2x}{8x^3 + 1}$
5. $\dfrac{6x^2 + 18x}{x^3 - 27}$
6. $\dfrac{3x^3 - 5x^2 + x - 3}{x^4 - 1}$
7. $\dfrac{12 + 13x - 2x^2}{8x^3 - 27}$
8. $\dfrac{2x^3 + 2x^2 + 10}{x^4 + x^2 + 1}$
9. $\dfrac{ax^2 - x^3}{x^3 + a^3}$
10. $\dfrac{x^3 - 6x^2 - 12x - 24}{x^4 - 16}$
11. $\dfrac{6x^2 - 5x - 11}{(3x + 1)(x^2 - 3)}$
12. $\dfrac{x^3 - 5x^2 - 1}{(x^2 + 4)(x^2 - 2x + 1)}$

Note:

We must only proceed if we have a proper fraction. Let's revisit our first example. We combined $\frac{2}{x-1}+\frac{3x}{x+1}$ to get $\frac{3x^2-x+2}{x^2-1}$. Since $\frac{3x^2-x+2}{x^2-1}$ is not a proper fraction we must divide the denominator into the numerator to obtain $3+\frac{5-x}{x^2-1}$. Using the decomposition process heretofore discussed we find that $3+\frac{5-x}{x^2-1}$ is equal to $3+\frac{2}{x-1} - \frac{3}{x+1}$ or if we combine the first and the last terms we will obtain $\frac{2}{x-1}+\frac{3x}{x+1}$.