Binomial Series

$$(1 + x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k$$

m is any real number and $|x| < 1$.

Example:

$$\frac{1}{\sqrt[5]{32 - x}} = \frac{1}{2}\left(1 - \frac{x}{32}\right)^{-\frac{1}{5}}$$

$$= \frac{1}{2}\left[\sum_{k=0}^{\infty} \binom{\frac{-1}{5}}{k} \left(-\frac{x}{32}\right)^k\right] = \frac{1}{2}\left[\sum_{k=0}^{\infty} \binom{\frac{-1}{5}}{k} (-1)^k \left(\frac{x}{32}\right)^k\right]$$

$$= \frac{1}{2}\left[\sum_{k=0}^{\infty} \binom{\frac{-1}{5}}{k} (-1)^k \frac{x^k}{2^{5k}}\right]$$

$$= \frac{1}{2}\left[1 + \left(-\frac{1}{5}\right)\left(\frac{x}{2^5}\right) + \frac{\left(-\frac{1}{5}\right)\left(-\frac{6}{5}\right)}{2!}\left(\frac{x^2}{2^{10}}\right) + \frac{\left(-\frac{1}{5}\right)\left(-\frac{6}{5}\right)\left(-\frac{11}{5}\right)}{3!}\left(\frac{x^3}{2^{15}}\right) + \cdots\right]$$

$$= \frac{1}{2} + \left(-\frac{1}{5 \cdot 2^6}\right)x + \left(-\frac{1 \cdot 6}{5^2 \cdot 2! \cdot 2^{11}}\right)x^2 + \left(-\frac{1 \cdot 6 \cdot 11}{5^3 \cdot 3! \cdot 2^{16}}\right)x^3 + \ldots$$

$$= \frac{1}{2} + \sum_{k=1}^{\infty} \frac{1 \cdot 6 \cdot \cdots (5k - 4)}{5^k \cdot 2^{5k+1} \cdot k!} x^k$$

Example:

$$\frac{1}{\sqrt{1-x^2}} = \left(1+\left(-x^2\right)\right)^{-\frac{1}{2}}$$

$$= 1 + \left(\frac{1}{2}\right)\left(-x^2\right) + \left(\frac{\left(\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\right)\left(-x^2\right)^2 + \left(\frac{\left(\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\right)\left(-x^2\right)^3 + \cdots$$

$$= 1 + \sum_{k=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2k-1)}{2^k\cdot k!}\, x^{2k}$$

Now arcsin(x) = $\int \frac{1}{\sqrt{1-x^2}}$**, so arcsin($x$) =**

$$\int 1 + \sum_{k=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2k-1)}{2^k\cdot k!}\, x^{2k} \quad dx$$

$$= x + \sum_{k=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2k-1)}{(2k+1)\cdot 2^k\cdot k!}\, x^{2k}$$