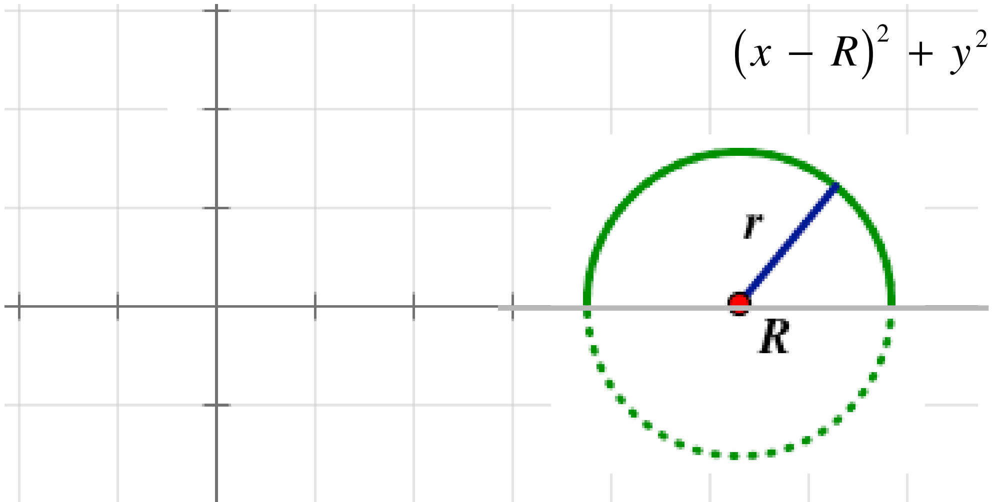
Surface Area of Torus



We will spin only the top semicircle so remember to **double your final answer.**

$$y = \sqrt{r^2 - (x - R)^2}$$

$$y' = \frac{-(x - R)}{\sqrt{r^2 - (x - R)^2}}$$

$$(y')^2 = \frac{(x - R)^2}{r^2 - (x - R)^2}$$

$$S = 2\pi \int_{R-r}^{R+r} x \sqrt{1 + \frac{(x - R)^2}{r^2 - (x - R)^2}}\, dx$$

$$= 2\pi \int_{R-r}^{R+r} x \sqrt{\frac{r^2}{r^2 - (x - R)^2}}\, dx$$

$$= 2\pi r \int_{R-r}^{R+r} \frac{x}{\sqrt{r^2 - (x - R)^2}}\, dx$$

Let $u^2 = (x - R)^2$ or $u = x - r$, $du = dx$ and $x = u + r$.

We now have $2\pi r \int_{-r}^{r} \frac{u + R}{\sqrt{r^2 - u^2}}\, du = 2\pi r \int_{-r}^{r} \frac{u}{\sqrt{r^2 - u^2}} + \frac{R}{\sqrt{r^2 - u^2}} du$

$$2\pi r\int_{-r}^{r}\frac{u}{\sqrt{r^2-u^2}} = 0$$

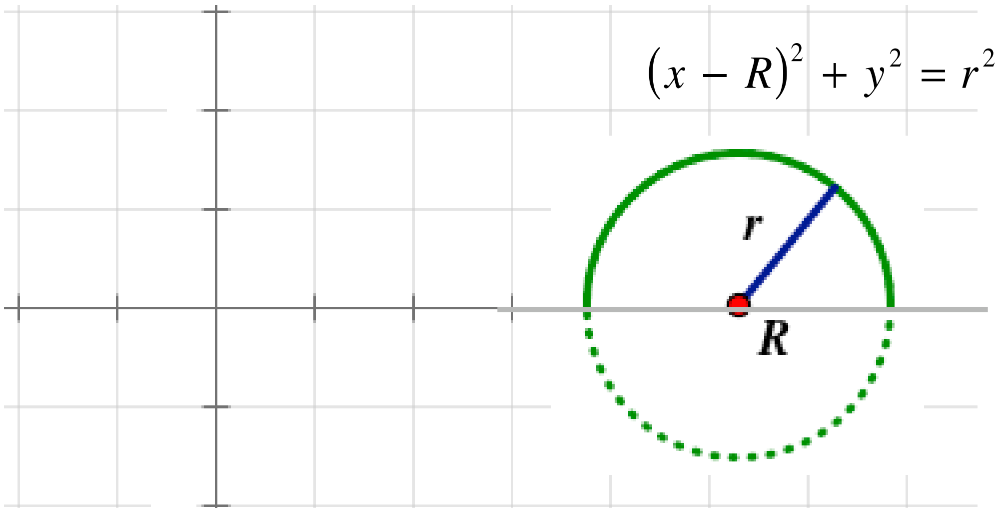
so $2\pi r\int_{-r}^{r}\frac{R}{\sqrt{r^2-u^2}}du = 2\pi rR\int_{-r}^{r}\frac{du}{\sqrt{r^2-u^2}}$. Let $u = r\sin\theta,\ du = r\cos\theta$

This gives us $2\pi rR\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{r\cos\theta}{\sqrt{r^2-r^2\sin^2\theta}}\,d\theta$

$$= 2\pi rR\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\cos\theta}{\sqrt{1-\sin^2\theta}}\,d\theta = 2\pi rR\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\cos\theta}{\cos\theta}\,d\theta = 2\pi^2 rR$$

Doubling we have $S = 4\pi^2 rR$

Volume of Torus



We will spin only the top semicircle so remember to **double your final answer.**

$$V = 2\pi \int_{R-r}^{R+r} x \sqrt{r^2 - (x - R)^2}\, dx$$

Let $u^2 = (x - R)^2$, $u = x - R$, $du = dx$, $x = u + R$

$$\Rightarrow V = 2\pi \int_{-r}^{r} (u + R) \sqrt{r^2 - u^2}\, du$$

$$= 2\pi \int_{-r}^{r} u \sqrt{r^2 - u^2}\, du + 2\pi \int_{-r}^{r} R \sqrt{r^2 - u^2}\, du$$

$$= 0 + 2\pi R \int_{-r}^{r} \sqrt{r^2 - u^2}\, du$$

$$= 2\pi R \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r\cos\theta \sqrt{r^2 - r^2 \sin^2\theta}\, d\theta$$

$$= 2\pi R r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta \sqrt{1 - \sin^2\theta}\, d\theta$$

$$= 2\pi R r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\, d\theta$$

$$= \pi^2 R r^2$$

Remember to double! $V = 2\pi^2 R r^2$