Power/Taylor series problems from the web ssite of Marta Hidegkuti.

(http://www.teaching.martahidegkuti.com/shared/lnotes/calculus.html)

Power Series:

1. Find all values of x for which the series converges. For what values will the series converge conditionally?

   a) $\displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{2n}}{n}$ b) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1}$ c) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\frac{x^{n}}{n!}$ d) $\displaystyle\sum_{n=1}^{\infty}\frac{1+2+3+\ldots+n}{1^2+2^2+3^2+\ldots+n^2}x^n$

2. Find a power series that converges to $\ln(1-x)$.

3. a) Find a series that converges to $\dfrac{1}{1+x^2}$.

   b) Find a series that converges to $\tan^{-1}x$.

4. Find the power series that converges to $\dfrac{1}{1+x}$ and use that to find the power series that converges to $\ln(1+x)$.

5. Consider $\displaystyle\sum_{n=0}^{\infty}\frac{n^2}{2^n}$

   a) Start with $\dfrac{1}{1-x}=1+x+x^2+x^3+\ldots..$ on $(-1,1)$ and differentiate both sides and multiply by x

   b) differentiate both sides and multiply by x again

   c) Set $x=\dfrac{1}{2}$

6. Find the radius of convergence. Then, within the radius of convergence, find the function to which the power series converges.

   a) $\displaystyle\sum_{n=0}^{\infty}3^n x^n$ b) $\displaystyle\sum_{n=0}^{\infty}(\ln x)^n$

7. Use power series to find the sum of the series $\displaystyle\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)^n}{n}$

Taylor Series:

1. Find the Taylor series generated by each of the following functions.

   a) $f(x) = x \sin x$ b) $f(x) = e^{3x}$ c) $f(x) = x^2 \tan^{-1} x$ d) $f(x) = \sinh x$

2. Use a Taylor series to determine the sum of each of the following series.

   a) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \ldots\ldots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$

   b) $1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \ldots\ldots = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!}$

   c) $\frac{\pi}{3} - \frac{\pi^3}{3^3 \cdot 3!} + \frac{\pi^5}{3^5 \cdot 5!} - \frac{\pi^7}{3^7 \cdot 7!} + \ldots\ldots = \sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+1}}{3^{2n+1}(2n+1)!}$

   d) $-1 + 2x - 3x^2 + 4x^3 - 5x^4 \ldots\ldots = \sum_{n=1}^{\infty} (-1)^n n x^{n-1}$

   e) $1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \ldots\ldots = \sum_{n=0}^{\infty} \frac{x^n}{n+1}$

3. Use the Taylor series generated by $f(x) = x \tan^{-1} x$ to determine the value of $f^{(208)}(0)$.

4. Consider the Taylor series generated by $f(x) = e^x$.

   a) Let i be the complex number with $i^2 = -1$.Simplify the expressions i^2, i^3, i^4, i^5, i^6, ...

   b) Write the Taylor series generated by $f(x) = e^x$ and evaluate it at $x = i\theta$ to prove that $e^{i\theta} = \cos\theta + i \sin\theta$

   c) Use the result $e^{i\theta} = \cos\theta + i \sin\theta$ to compute $e^{i\pi}$