Numerical Series Behavior Tests

Zero Sequence Test

$$\{a_k\} \not\to 0 \quad \Rightarrow \sum_{k=0}^{\infty} a_k \text{ diverges}$$

Asymptotics

$$\ln x << x << x^a << a^x << n! << x^x$$

Example

$$\sum_{k=0}^{\infty} \frac{k^2+3k}{2k^2+5}$$

Numerical Series Behavior Tests

Grandi's Series

$$\sum_{k=0}^{\infty} (-1)^k \text{ diverges}$$

Alternating Series Test

$$\{a_k\} \to 0$$

$$\{a_k\} \text{ nonincreasing with positve terms}$$

$$\Rightarrow \sum_{k=0}^{\infty} (-1)^k a_k \text{ converges}$$

Example

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k-2}{k^2+3k}$$

Warning !

If a series fails the alternating series test, it does not necessarily diverge !

Example

$$\sum_{k=0}^{\infty} (-1)^k a_k \qquad \text{where } \{a_k\} = \begin{cases} 2^{-k} & \text{for } k \text{ even} \\ 3^{-k} & \text{for } k \text{ odd} \end{cases}$$

This alternating series does not diverge! It converges.

$$\sum_{k=0}^{\infty} \frac{1}{2^{(2k)}} - \frac{1}{3^{(2k+1)}} = \sum_{k=0}^{\infty} (-1)^k a_k \qquad \{a_k\} = \begin{cases} 2^{-k} & \text{for } k \text{ even} \\ 3^{-k} & \text{for } k \text{ odd} \end{cases}$$

$$\frac{1}{3^1} \quad \frac{1}{2^2} \quad \frac{1}{3^3} \quad \frac{1}{2^4} \quad \frac{1}{3^5} \quad \frac{1}{2^6} \quad \frac{1}{3^7} \quad \frac{1}{2^8}$$

$$a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6 \quad a_7 \quad a_8$$

$$\frac{1}{3} \quad \frac{1}{4} \quad \frac{1}{27} \quad \frac{1}{16} \quad \frac{1}{243} \quad \frac{1}{64} \quad \frac{1}{747} \quad \frac{1}{256}$$

Numerical Series Behavior Tests

$$\sum_{k=0}^{\infty} \left| a_k \right| \text{ converges } \Rightarrow \sum_{k=0}^{\infty} a_k \text{ converges}$$

Ratio Test

$$\lim_{k \to \infty} \left| \frac{a_k + 1}{a_k} \right| < 1$$

Example

$$\sum_{k=1}^{\infty} \frac{k^3}{k!}$$

Numerical Series Behavior Tests

Root Test

$$\lim_{k \to \infty} \sqrt[k]{|a_k|} < 1$$

Example

$$\sum_{k=1}^{\infty} \frac{k^4}{2^k}$$

Warning !

If in the ratio/root test, **the limit is equal to 1**, the test is inconclusive and we must proceed to some other convergence criterion.

Example

$$\sum_{k=1}^{\infty} \frac{1}{k} \qquad\qquad \sum_{k=1}^{\infty} \frac{1}{k^2}$$

Just look at the two harmonic series above. Both will yield a limit of 1 by the rato test, yet one converges and the other diverges.

Numerical Series Behavior Tests

Asymptotics

Integral Test

$\{a_k\}$ has nonnegative terms

f nonincreasing with $f(k) = a_k$

then $\sum_{k=0}^{\infty} a_k$ *and* $\int_1^{\infty} f(x)\,dx$ behave the same

Example

$$\sum_{k=1}^{\infty} \frac{\ln k}{k}$$

Numerical Series Behavior Tests

Asymptotics

Dominating Series Tests

$\sum_{k=0}^{\infty} a_k$, $\sum_{k=0}^{\infty} b_k$ series with nonnegative terms

- if $\sum_{k=0}^{\infty} b_k$ converges and $a_k \le b_k$, then

  $\sum_{k=0}^{\infty} a_k$ converges

$\sum_{k=0}^{\infty} a_k$, $\sum_{k=0}^{\infty} b_k$ series with nonnegative terms

- if $\sum_{k=0}^{\infty} b_k$ diverges and $a_k \ge b_k$, then

  $\sum_{k=0}^{\infty} a_k$ diverges

Numerical Series Behavior Tests

Asymptotics

Limit Comparison Tests

$\sum_{k=0}^{\infty} a_k$, $\sum_{k=0}^{\infty} b_k$ series with nonnegative terms

1) if $\frac{a_k}{b_k} = c \geq 0$ and $\sum_{k=0}^{\infty} b_k$ converges

then $\sum_{k=0}^{\infty} a_k$ converges

2) if $\frac{a_k}{b_k} = c > 0$ or ∞ and $\sum_{k=0}^{\infty} b_k$ diverges

then $\sum_{k=0}^{\infty} a_k$ diverges

Example

$$\sum_{k=0}^{\infty} \frac{1}{3^k - k}$$

Numerical Series Behavior Tests

Telescoping Series

Example

$$\sum_{k=0}^{\infty} \frac{1}{k^2 + 3k + 2}$$