Find the Maclaurin series for arctan x and test for convergence.

$$f(x) = \sum_{k=0}^{\infty} \left(\frac{f^{(k)}(0)}{k!} \right) x^k$$

$$k = 0: \rightarrow \arctan x \rightarrow \arctan 0 \rightarrow 0 \rightarrow \frac{0}{0!} \rightarrow 0 \cdot x^0 \rightarrow 0$$

$$k = 1: \rightarrow \frac{1}{1 + x^2} \rightarrow \frac{1}{1 + (0)^2} \rightarrow 1 \rightarrow \frac{1}{1!} \rightarrow 1 \cdot x^1 \rightarrow x$$

$$k = 2: \rightarrow -\frac{2x}{\left(1 + x^2\right)^2} \rightarrow -\frac{2 \cdot 0}{\left(1 + 0^2\right)^2} \rightarrow 0 \rightarrow \frac{0}{2!} \rightarrow 0$$

$$k = 3: \rightarrow \frac{8x^2}{\left(1 + x^2\right)^3} - \frac{2}{\left(1 + x^2\right)^2} \rightarrow -2 \rightarrow \frac{-2}{3!} \rightarrow -\frac{2}{3!} \cdot x^3$$

$$k = 4: \rightarrow \frac{48x^3}{\left(1 + x^2\right)^4} + \frac{24x}{\left(1 + x^2\right)^3} \rightarrow 0 \rightarrow \frac{0}{4!} \cdot x^4 \rightarrow 0$$

$$k = 5: \rightarrow \frac{384x^4}{\left(1 + x^2\right)^5} - \frac{288x^2}{\left(1 + x^2\right)^4} + \frac{24}{\left(1 + x^2\right)^3} \rightarrow 24 \rightarrow \frac{24}{5!} \rightarrow \frac{24}{5!} \cdot x^5$$

$$k = 6: \rightarrow -\frac{3840x^5}{\left(1 + x^2\right)^6} + \frac{3840x^3}{\left(1 + x^2\right)^5} - \frac{720x^3}{\left(1 + x^2\right)^4} \rightarrow 0 \rightarrow \frac{0}{6!} \cdot x^6 \rightarrow 0$$

$$\arctan x = x - \frac{2}{3!}x^3 + \frac{24}{5!}x^5 + \ldots = \arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \ldots$$

$$\arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$$

Now check for convergence.

$$\lim_{k\to\infty}\left|\frac{\dfrac{x^{2k+3}}{2k+2}}{\dfrac{x^{2k+1}}{2k+1}}\right| = \lim_{k\to\infty}\left|\frac{x^{2k+3}}{2k+2}\cdot\frac{2k+1}{x^{2k+1}}\right| = \lim_{k\to\infty}\left|\frac{(2k+1)}{(2k+2)}\cdot x^2\right| = x^2\cdot 1$$

$\therefore$ we have convergence on $(-1, 1)$

Check for x = -1 :

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}(-1)^{2k+1} = \sum_{k=0}^{\infty}\frac{(-1)^{3k+1}}{2k+1} \text{ converges by alternating series test}$$

Check for x = 1 :

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}(1)^{2k+1} = \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1} \text{ converges by alternating series test}$$

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}x^{2k+1} \text{ converges to } \arctan x \text{ on } [-1, 1]$$

Using just the fact that

$$\sum_{k=0}^{\infty} x^k \text{ converges to } \frac{1}{1-x} \text{ on } (-1, 1)$$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} \leftarrow \sum_{k=0}^{\infty}(-x^2)^k$$

$$\frac{1}{1+x^2} = \sum_{k=0}^{\infty}(-x^2)^k = 1 - x^2 + x^4 - x^6 + x^8 - \ldots$$

$$\int \frac{1}{1+x^2}dx = \int 1 - x^2 + x^4 - x^6 + x^8 - \ldots dx$$

$$\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}\, x^{2k+1}$$

This is called *Gregory's Series* and can be used for the calculation of π.

Since

$$\frac{\pi}{4} = \arctan 1 = 1 - \frac{1}{3}1^3 + \frac{1}{5}1^5 - \frac{1}{7}1^7 + \frac{1}{9}1^9 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9}\cdots$$

$$\text{or } \pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9}\ldots\right)$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots$$

Pi by alternating sums
4
2.66667
3.46667
2.89524
3.33968
2.97605
3.28374
3.01707
3.25237
3.04184
3.23232
3.0584
3.2184
3.07025
3.20819
3.07915
3.20037
3.08608
3.19419
3.09162
3.18918
3.09616
3.18505
3.09994
3.18158