Derivatives of Inverse Trig Functions

Using the formula for calculating the derivative of inverse functions

$$\left(f^{-1}\right)' = \frac{1}{f'\left(f^{-1}\right)}$$

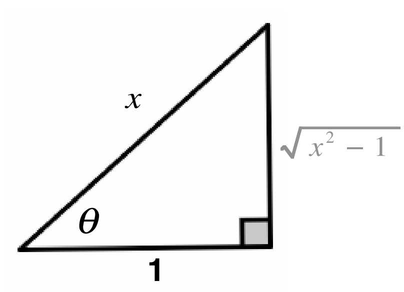
we have shown that $\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1 - x^2}}$ and $\frac{d}{dx}(\arctan x) = \frac{1}{1 + x^2}$.

To complete the list of derivatives of the inverse trig functions, I will show how to find $\frac{d}{dx}(\operatorname{arc}\sec x)$.

Let $f(x) = \sec x$ and $f^{-1}(x) = \operatorname{arcsec} x$. We know $f'(x) = \sec x \tan x$, so by the formula above we have

$$(\operatorname{arc}\sec x)' = \frac{1}{\sec(\operatorname{arcsec} x)\tan(\operatorname{arcsec} x)}$$

which directs us to consider the following -

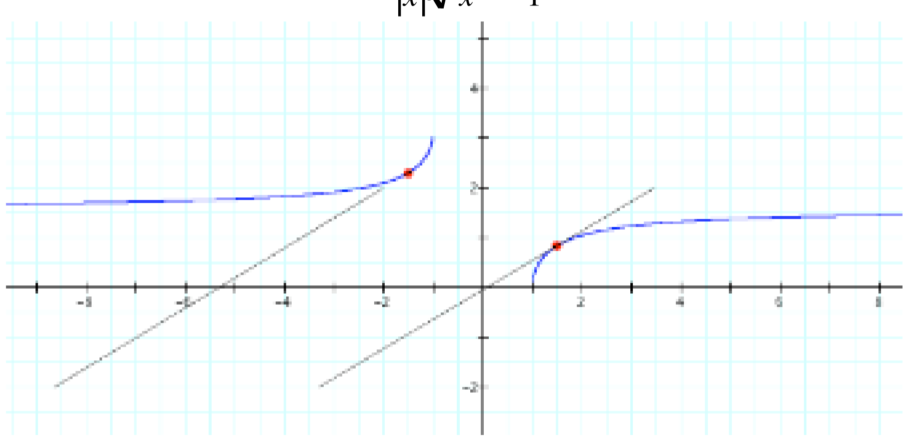


$\theta = \operatorname{arcsec} x$, $\sec(\operatorname{arc}\sec x) = x$ and $\tan(\operatorname{arcsec} x) = \sqrt{x^2 - 1}$. So

$$(\operatorname{arc}\sec x)' = \frac{1}{x\sqrt{x^2 - 1}}$$

But the graph of $y = \operatorname{arcsec} x$ on the below shows the derivative of this function is always positive. To accommodate this fact we have

$$(\operatorname{arc}\sec x)' = \frac{1}{|x|\sqrt{x^2 - 1}}$$



All other derivatives of the inverse trig functions are similarly derived. Giving us the list below.

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1 - x^2}} \qquad \frac{d}{dx}(\operatorname{arc\,csc} x) = \frac{-1}{|x|\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}(\arccos x) = \frac{-1}{\sqrt{1 - x^2}} \qquad \frac{d}{dx}(\operatorname{arc\,sec} x) = \frac{1}{|x|\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}(\arctan x) = \frac{1}{1 + x^2} \qquad \frac{d}{dx}(\operatorname{arccot} x) = \frac{-1}{1 + x^2}$$

Derivatives of Inverse Hyperbolic Functions

Unfortunately, since we cannot easily reckon hyperbolic trig functions with right triangles, we calculate the derivatives of these functions differently. We use the pre calculus technique of solving for rules of correspondence of inverse functions. For example,

Let $y = \operatorname{arcsinh} x$ and let us solve for x in terms of y.

Since $y = \operatorname{arcsinh} x$, then $x = \sinh y$ or equivalently

$$x = \frac{1}{2}e^y - \frac{1}{2}e^{-y}$$

$$2x = e^y - e^{-y} = e^y - \frac{1}{e^y} = \frac{e^{2y} - 1}{e^y}$$ which means that

$$2xe^y = e^{2y} - 1$$ or that

$$e^{2y} - 2xe^y - 1 = 0$$ which is a quadratic in e^y.

Using the quadratic formula, $a = 1$, $b = -2x$, and $c = -1$. We have

$$e^y = \frac{2x + \sqrt{4x^2 + 4}}{2} = x + \sqrt{x^2 + 1}.$$ Solving for y

$$y = \ln\left(x + \sqrt{x^2 + 1}\right).$$ We now have

$$y = \operatorname{arcsinh} x = \ln\left(x + \sqrt{x^2 + 1}\right).$$ Now we can use the chain rule

to find $\frac{d}{dx}(\text{arcsinh } x) = \frac{d}{dx}\left(\ln\left(x + \sqrt{x^2 + 1}\right)\right)$

$$\left(\ln\left(x + \sqrt{x^2 + 1}\right)\right)' =$$

$$\frac{1}{x + \sqrt{x^2 + 1}} \cdot \left(x + \sqrt{x^2 + 1}\right)' =$$

$$\frac{1}{x + \sqrt{x^2 + 1}} \cdot \left(1 + \frac{1}{2\sqrt{x^2 + 1}}\right) \cdot 2x =$$

$$\frac{1}{x + \sqrt{x^2 + 1}} \cdot \left(x + \sqrt{x^2 + 1}\right)' =$$

$$\frac{1}{x + \sqrt{x^2 + 1}} \cdot \left(\frac{\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1}}\right) =$$

$$\frac{1}{\sqrt{x^2 + 1}}\ .\ \text{So}$$

$$\frac{d}{dx}(\text{arcsinh } x) = \frac{1}{\sqrt{x^2 + 1}}$$

All other derivatives of the inverse hyperbolic trig functions are similarly derived. Giving us the list below.

$$\frac{d}{dx}(\text{arcsinh } x) = \frac{1}{\sqrt{x^2 + 1}}$$

$$\frac{d}{dx}(\text{arccosh } x) = \frac{1}{\sqrt{x^2 - 1}},\ x > 1$$

$$\frac{d}{dx}(\text{arctanh } x) = \frac{1}{1 - x^2},\ |x| < 1$$

$$\frac{d}{dx}(\text{arccsch } x) = \frac{-1}{|x|\sqrt{1 + x^2}}$$

$$\frac{d}{dx}(\text{arcsech } x) = \frac{-1}{x\sqrt{1 - x^2}},\ 0 < x < 1$$

$$\frac{d}{dx}(\text{arccoth } x) = \frac{1}{1 - x^2},\ |x| > 1$$