Infinitesimal Calculus

$\Delta x \Delta y$ and $\dfrac{\Delta y}{\Delta x}$ "cannot stand"

• Derivative of the sum/difference of two functions

$(x+\Delta x) \pm (y+\Delta y) = (x+y)+\Delta x+\Delta y$

$\therefore$ we have a change of $\Delta x+\Delta y$.

• Derivative of the product of two functions

$(x+\Delta x)(y+\Delta y) = xy+\Delta xy+x\Delta y+\Delta x\Delta y$

$\therefore$ we have a change of $\Delta xy+x\Delta y$.

• Derivative of the product of three functions

$$(x+\Delta x)(y+\Delta y)(z+\Delta z) = xyz+\Delta xyz+x\Delta yz+xy\Delta z+ x\Delta y\Delta z+\Delta x\Delta yz+x\Delta y\Delta z+\Delta x\Delta y\Delta$$

$\therefore$ we have a change of $\Delta xyz+x\Delta yz+xy\Delta z$.

• Derivative of the quotient of two functions

Let $u=\dfrac{x}{y}$. Then by the product rule above, $yu=x$ yields

$u\Delta y+y\Delta u=\Delta x$. Substituting for u its value, we have

$\dfrac{x\Delta y}{y}+y\Delta u=\Delta x$. Finding the value of Δu, we have $\dfrac{\Delta xy-x\Delta y}{y^2}$

• Derivative of a power function (and the "chain rule")

Let $y=x^m$. $\therefore$ $y=x\cdot x\cdot x\cdot\ldots\cdot x$ (m times). By a generalization of the product rule,

$\Delta y=(x^{m-1}\Delta x)(x^{m-1}\Delta x)(x^{m-1}\Delta x)\ldots\cdot(x^{m-1}\Delta x)$ m times.

$\therefore$ we have $\Delta y = mx^{m-1}\Delta x$.

• Derivative of the logarithmic function

Let $y = x^n$, n being constant. Then $\log y = n \log x$. Differentiating $y = x^n$, we have

$dy = nx^{n-1}dx$, or $n = \dfrac{dy}{x^{n-1}dx} = \dfrac{dy}{\frac{y}{x}dx} = \dfrac{\frac{dy}{y}}{\frac{dx}{x}}$, since $x^{n-1} = \dfrac{y}{x}$. Again, whatever the differentials of $\log x$ and $\log y$ are, we have $d(\log y) = n \cdot d(\log x)$, or $n = \dfrac{d(\log y)}{d(\log x)}$. Placing these values of n equal to each other, we obtain $\dfrac{d(\log y)}{d(\log x)} = \dfrac{\frac{dy}{y}}{\frac{dx}{x}}$. Now let m be the factor by which $\dfrac{dy}{y}$ must be multiplied to make it equal to $d(\log y)$, then is $d(\log x) = \dfrac{mdx}{x}$.

We are now to show that m is a constant depending upon the base of the system.

To do this take $y = x^n$, from which we find as before $n' = \dfrac{d(\log y)}{d(\log x)} = \dfrac{\frac{dy}{y}}{\frac{dx}{x}}$.

But m is the ratio of $d(\log y)$ to $\dfrac{dy}{y}$; hence $d(\log z) = \dfrac{mdz}{z}$. Thus we see that in any case the same ratio exists between the differential of the logarithm of a number divided by the number. Therefore m is a constant factor. To show that m depends upon the base of the system we have to recur to the definition of a logarithm to see that the only quantities involve are *the number*, its *logarithm*, and the *base* of the system. Of these the two former are variable, whence, as the base is the only constant in the scheme, m is a function of the base.

Finally, as m depends upon the base of the system, the base may be so taken that $m = 1$. The system of logarithms founded on this base is called the *Napierian system* and denoted by $y = \ln x$, $\therefore (\ln x)' = \dfrac{dx}{x}$.

• Derivative of the exponential function

Let $y = a^x$. Taking the logarithms of both members $\log y = x \log a$. Differentiating $\dfrac{mdy}{y} \log a \ dx$, or $dy = \dfrac{a^x \log a \ dx}{m}$, remembering that $y = a^x$, and that $\log a$ is constant.

• **Derivative of the sine function**

Let *x* be any arc (or angle) and y be its sine, *i.e.* let $y = \sin x$. If *x* takes an infinitesimal increment (*dx*), let *dy* represent the contemporaneous infinitesimal increment of *y*. The the consecutive state of the function is

$$y + dy = \sin(x + dx) = \sin x \cos dx + \sin dx \cos x.$$

Now cos *dx* = 1, since as an angle grow less its cosin approaches the radius in value, and *at the limit*, is the radius. Moreover, as an angle grows less the sine and the corresponding arc approach equality, and *at the limit* we have $\sin dx = dx$. Thus the consecutive state may therefore be written $y + dy = \sin x + \cos x dx$.

From this subtract $y = \sin x$

and we have $dy = \cos x dx$.

• **Derivative of the cosine function**

Let *x* be any arc (or angle) and y be its cosine, *i.e.* let $y = \cos x$. Since $\cos x = \sin(90^\circ - x)$ we have $y = \sin(90^\circ - x)$. Differentiating this by the preceding proposition, we obtain $dy = \cos(90^\circ - x)(90^\circ - x)' = \cos(90^\circ - x)(-dx) = -\sin x dx$.

It was known that $\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$. Also known was $\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6}$. Here we can see that the derivative of sin*x* is cos*x* and the derivative of cos*x* is the negative sin*x*.